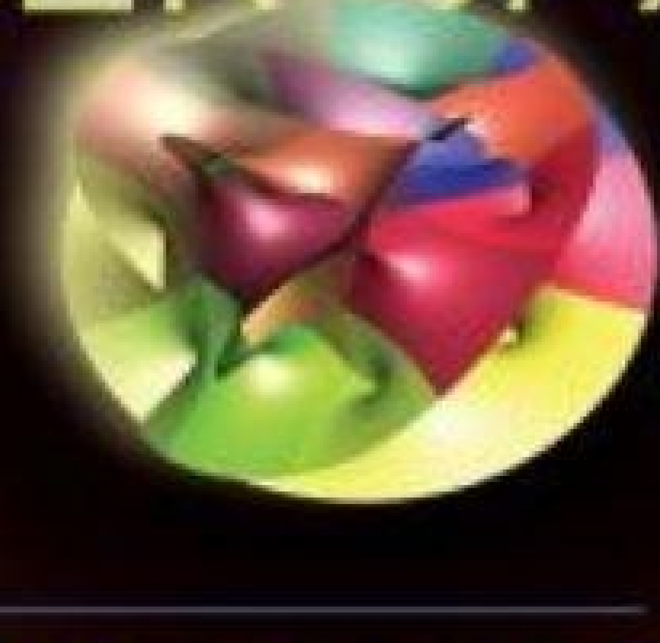
STRING THEORY and the GEOMETRY of the UNIVERSE'S HIDDEN DIMENSIONS

THE SHAPE OF INNER SPACE



SHING-TUNG YAU

WINNER OF THE FIELDS MEDAL

and STEVE NADIS

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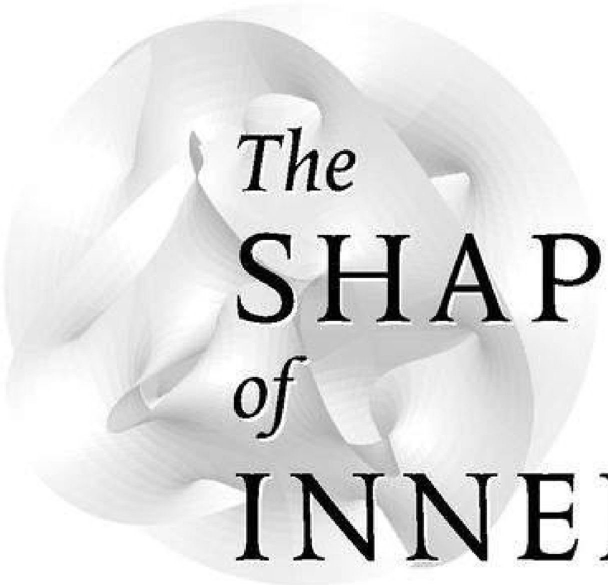

The SHAPE *of* INNER SPACE

STRING THEORY AND THE GEOMETRY OF THE UNIVERSE'S HIDDEN DIMENSIONS

Shing-Tung Yau

and

Steve Nadis

Illustrations by
Xianfeng (David) Gu and Xiaotian (Tim) Yin

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SPACE/TIME

Time, time
why does it vanish?
All manner of things
what infinite variety.
Three thousand rivers
all from one source.
Time, space
mind, matter, reciprocal.
Time, time
it never returns.
Space, space
how much can it hold?
In constant motion
always in flux.
Black holes lurking
mysteries afoot.
Space and time
one without bounds.
Infinite, infinite
the secrets of the universe.
Inexhaustible, lovely
in every detail.
Measure time, measure space
no one can do it.
Watched through a straw
what's to be learned has no end.

SHING-TUNG YAU
BEIJING, 2002

PREFACE

Mathematics is often called the language of science, or at least the language of the physical science and that is certainly true: Our physical laws can only be stated precisely in terms of mathematic equations rather than through the written or spoken word. Yet regarding mathematics as merely language doesn't do justice to the subject at all, as the word leaves the erroneous impression that, sav for some minor tweaks here and there, the whole business has been pretty well sorted out.

In fact, nothing could be further from the truth. Although scholars have built a strong foundatio over the course of hundreds—and indeed thousands—of years, mathematics is still very much thriving and dynamic enterprise. Rather than being a static body of knowledge (not to suggest th languages themselves are set in stone), mathematics is actually a dynamic, evolving science, with ne insights and discoveries made every day rivaling those made in other branches of science, thoug mathematical discoveries don't capture the headlines in the same way that the discovery of a ne elementary particle, a new planet, or a new cure for cancer does. In fact, save for the proof of centuries-old problem from time to time, they rarely capture headlines at all.

Yet for those who appreciate the sheer force of mathematics, it can be viewed as not just a languag but as the surest path to the truth—the bedrock upon which the whole edifice of physical science rest The strength of this discipline, again, lies not simply in its ability to explain physical reality or reveal it, because to a mathematician, mathematics is reality. The geometric figures and spaces, who existence we prove, are just as real to us as are the elementary particles of physics that make up a matter. But we consider mathematical structures even more fundamental than the particles of natu because mathematical structures can be used not only to understand such particles but also understand the phenomena of everyday life, such as the contours of a human face or the symmetry flowers. What excites geometers perhaps most of all is the power and beauty of the abstract principl that underlie the familiar forms and shapes of our contemporary world.

For me, the study of mathematics and my specialty, geometry, has truly been an adventure. I sti recall the thrill I felt during my first year of graduate school, when—as a twenty-year-old fresh off th boat, so to speak—I first learned about Einstein's theory of gravity. I was struck by the notion th gravity and curvature could be regarded as one and the same, as I'd already become fascinated wi curved surfaces during my undergraduate years in Hong Kong. Something about these shapes appeal to me on a visceral level. I don't know why, but I couldn't stop thinking about them. Hearing th curvature lay at the heart of Einstein's theory of general relativity gave me hope that someday, and some way, I might be able to contribute to our understanding of the universe.

The Shape of Inner Space describes my explorations in the field of mathematics, focusing on o discovery in particular that has helped some scientists build models of the universe. No one can sa for sure whether these models will ultimately prove correct. But the theory underlying these mode nevertheless, possesses a beauty that I find undeniable.

Taking on a book of this nature has been challenging, to say the least, for someone like me who more comfortable with geometry and nonlinear differential equations than writing in the Englis language, which is not my native tongue. I find it frustrating because there's a great clarity, as well a a kind of elegance, in mathematical equations that is difficult, if not impossible, to express in word It's a bit like trying to convey the majesty of Mount Everest or Niagara Falls without any pictures.

Fortunately, I've gotten some well-needed help on this front. Although this narrative is told throug my eyes and in my voice, my coauthor has been responsible for translating the abstract and abstru mathematics into (hopefully) lucid prose.

When I proved the Calabi conjecture—an effort that lies at the heart of this book—I dedicated th paper containing that proof to my late father, Chen Ying Chiu, an educator and philosopher wh instilled in me a respect for the power of abstract thought. I dedicate this book to him and to my la mother, Leung Yeuk Lam, both of whom had a profound influence on my intellectual growth. I addition, I want to pay tribute to my wife, Yu-Yun, who has been so tolerant of my rather excessiv (and perhaps obsessive) research and travel schedule, and to my sons, Isaac and Michael, of whom I' very proud.

I also dedicate this book to Eugenio Calabi, the author of the aforementioned conjecture, whom I'v known for nearly forty years. Calabi were an enormously original mathematician with whom I've bee linked for more than a quarter century through a class of geometric objects, Calabi-Yau manifold which serve as the principal subject of this book. The term *Calabi-Yau* has been so widely used sin it was coined in 1984 that I almost feel as if Calabi is my first name. And if it is to be my first name- at least in the public's mind—it's one I'm proud to have.

The work that I do, much of which takes place along the borders between mathematics an theoretical physics, is rarely done in isolation, and I have benefited greatly from interactions wi friends and colleagues. I'll mention a few people, among many, who have collaborated with m directly or inspired me in various ways.

First, I'd like to pay tribute to my teachers and mentors, a long line of illustrious people th includes S. S. Chern, Charles Morrey, Blaine Lawson, Isadore Singer, Louis Nirenberg, and th aforementioned Calabi. I'm pleased that Singer invited Robert Geroch to speak at a 1973 Stanfo conference, where Geroch inspired my work with Richard Schoen on the positive mass conjecture. M subsequent interest in physics-related mathematics has always been encouraged by Singer.

I'm grateful for the conversations I had on general relativity while visiting Stephen Hawking an Gary Gibbons at Cambridge University. I learned about quantum field theory from one of the maste of the subject, David Gross. I remember in 1981, when I was a professor at the Institute for Advanc Study, the time Freeman Dyson brought a fellow physicist, who had just arrived in Princeton, into m office. The newcomer, Edward Witten, told me about his soon-to-be-published proof of the positiv mass conjecture—a conjecture I had previously proved with a colleague using a very differe ~~technique. I was struck, for the first of many times to come, by the sheer force of Witten~~ mathematics.

Over the years, I’ve enjoyed close collaborations with a number of people, including Scho (mentioned above), S. Y. Cheng, Richard Hamilton, Peter Li, Bill Meeks, Leon Simon, and Kare Uhlenbeck. Other friends and colleagues who have added to this adventure in many ways inclu Simon Donaldson, Robert Greene, Robert Osserman, Duong Hong Phong, and Hung-Hsi Wu.

I consider myself especially lucky to have spent the past twenty-plus years at Harvard, which h provided an ideal environment for interactions with both mathematicians and physicists. During m time here, I’ve gained many insights from talking to Harvard math colleagues—such as Josep Bernstein, Noam Elkies, Dennis Gaitsgory, Dick Gross, Joe Harris, Heisuke Hironaka, Arthur Jaf (also a physicist), David Kazdhan, Peter Kronheimer, Barry Mazur, Curtis McMullen, Davi Mumford, Wilfried Schmid, Yum-Tong Siu, Shlomo Stern-berg, John Tate, Cliff Taubes, Richar Taylor, H. T. Yau, and the late Raoul Bott and George Mackey—while having memorable exchange with MIT math colleagues as well. On the physics side, I’ve had countless rewarding conversatio with Andy Strominger and Cumrun Vafa.

In the past ten years, I was twice an Eilenberg visiting professor at Columbia, where I had ma stimulating conversations with faculty members, especially with Dorian Goldfeld, Richard Hamilto Duong Hong Phong, and S. W. Zhang. I was also a Fairchild visiting professor and Moore visiti professor at Caltech, where I learned a lot from Kip Thorne and John Schwarz.

Over the last twenty-three years, I have been supported by the U.S. government through t National Science Foundation, the Department of Energy, and DARPA in my research related t physics. Most of my postdoctoral fellows received their Ph.D.s in physics, which is somewhat unusu in our discipline of mathematics. But the arrangement has been mutually beneficial, as they hav learned some mathematics from me and I have learned some physics from them. I am glad that ma of these postdoctoral fellows with physics backgrounds later became prominent professors mathematics departments at Brandeis, Columbia, Northwestern, Oxford, Tokyo, and oth universities. Some of my postdocs have done important work on Calabi-Yau manifolds, and many them have also helped on this book: Mboyo Esole, Brian Greene, Gary Horowitz, Shinobu Hoson Tristan Hubsch, Albrecht Klemm, Bong Lian, James Sparks, Li-Sheng Tseng, Satoshi Yamaguchi, an Eric Zaslow. Finally, my former graduate students—including Jun Li, Kefeng Liu, Melissa Li Dragon Wang, and Mu-Tao Wang—have made noteworthy contributions in this area as well, some which will be described in the pages to come.

—SHING-TUNG YAU, CAMBRIDGE, MASSACHUSETTS, MARCH 2010

Odds are I never would have known about this project were it not for Henry Tye, a Cornell physici (~~and a friend of Yau's), who suggested that my coauthor-to-be might steer me to an interesting~~ tale o two. Henry was right about this, as he has been about many other things. I'm grateful to him f helping to launch me on this unexpected journey and for assisting me at many junctures along th way.

As Yau has often said, when you venture down a path in mathematics, you never know where it wi end up. The same has been true on the writing end of things. The two of us pretty much agreed durin our very first meeting to write a book together, though it took a long while for us to know what th book would be about. In some ways, you might say we didn't really know that until the book w finished.

Now a few words about the product of this collaboration in an attempt to keep any confusion to minimum. My coauthor is, of course, a mathematician whose work is central to much of the sto related here. Sections of the book in which he was an active participant are generally written in th first person, with the "I" in this case referring to him and him alone. However, even though the boo has its fair share of personal narrative, this work should probably not be characterized as Yau autobiography or biography. That's because part of the discussion relates to people Yau doesn't kno (or who died long before he was born), and some of the subject matter described—such experimental physics and cosmology—lies outside his areas of expertise. These sections, which a written in a third-person voice, are largely based on interviews and other research I conducted.

While the book is, admittedly, an unusual blend of our different backgrounds and perspectives, seemed to be the best way for the two of us to recount a story that we both considered worth tellin The task of actually getting this tale down on paper relied heavily on my coauthor's extraordina grasp of numbers and hopefully profited as well from his collaborator's facility with words.

One other point on the issue of whether this ought to be regarded as an autobiography: Although th book certainly revolves around Yau's work, I would suggest that the main character is not Yau himse but rather the class of geometric shapes—so-called Calabi-Yau manifolds—that he helped invent.

Broadly speaking, this book is about understanding the universe through geometry. Gener relativity, a geometry-based description of gravity that has achieved stunning success in the pa century, offers one example. String theory represents an ambitious attempt to go even further, an geometry is vital to this quest, with six-dimensional Calabi-Yau shapes assuming a special place this theory. The book tries to present some of the ideas from geometry and physics needed understand where Calabi-Yau manifolds came from and why some physicists and mathematicia consider them important. The book focuses on various aspects of these manifolds—their definin features, the mathematics that led to their discovery, the reasons string theorists find them intriguin and the question of whether these shapes hold the key to our universe (and perhaps to other univers as well).

That, at least, is what *The Shape of Inner Space* is supposed to be about. Whether it lives up to th bill~~ing may be open to debate. But there is no doubt in my mind that this book would never have~~ con to fruition without technical, editorial, and emotional support from many people—too many, I' afraid, to list in full, but I will mention as many as I can.

I received a tremendous amount of help from people already singled out by my coauthor. Thes include Eugenio Calabi, Simon Donaldson, Brian Greene, Tristan Hubsch, Andrew Strominger, L Sheng Tseng, Cumrun Vafa, Edward Witten, and, most of all, Robert Greene, Bong Lian, and Li-Sher Tseng. The latter three provided me with math and physics tutorials throughout the writing proces exhibiting expository skills and levels of patience that boggle the mind. Robert Greene, in particula spoke with me a couple of times a week during busy stretches to guide me through thorny bits differential geometry. Without him, I would have been sunk—many times over. Lian got me started i thinking about geometric analysis, and Tseng helped out immensely with last-minute changes in o ever-evolving manuscript.

The physicists Allan Adams, Chris Beasley, Shamit Kachru, Liam McAllister, and Burt Ovru fielded questions from me at various times of day and night, carrying me through many a rough patc Other individuals who were exceedingly generous with their time include Paul Aspinwall, Melan Becker, Lydia Bieri, Volker Braun, David Cox, Frederik Denef, Robbert Dijkgraaf, Ron Donagi, Mik Douglas, Steve Giddings, Mark Gross, Arthur Hebecker, Petr Horava, Matt Kleban, Igor Klebano Albion Lawrence, Andrei Linde, Juan Maldacena, Dave Morrison, Lubos Motl, Hirosi Ooguri, Tor Pantev, Ronen Plesser, Joe Polchinski, Gary Shui, Aaron Simons, Raman Sundrum, Wati Taylor, Bre Underwood, Deane Yang, and Xi Yin.

That is merely the tip of the iceberg, as I've also received help from Eric Adelberger, Saleem Al Bruce Allen, Nima Arkani-Hamed, Michael Atiyah, John Baez, Thomas Banchoff, Katrin Becke George Bergman, Vincent Bouchard, Philip Candelas, John Coates, Andrea Cross, Lance Dixon, Davi Durlach, Dirk Ferus, Felix Finster, Dan Freed, Ben Freivogel, Andrew Frey, Andreas Gathmann Doron Gepner, Robert Geroch, Susan Gilbert, Cameron Gordon, Michael Green, Paul Green, Arthu Greenspoon, Marcus Grisaru, Dick Gross, Monica Guica, Sergei Gukov, Alan Guth, Robert S. Harri Matt Headrick, Jonathan Heckman, Dan Hooper, Gary Horowitz, Stanislaw Janeczko, Lizhen J Sheldon Katz, Steve Kleiman, Max Kreuzer, Peter Kronheimer, Mary Levin, Avi Loeb, Feng Lu Erwin Lutwak, Joe Lykken, Barry Mazur, William McCallum, John McGreevy, Stephen Miller, Clif Moore, Steve Nahn, Gail Oskin, Rahul Pandharipande, Joaquín Pérez, Roger Penrose, Miles Rei Nicolai Reshetikhin, Kirill Saraikin, Karen Schaffner, Michael Schulz, John Schwarz, Ashoke Se Kris Snibbe, Paul Shellard, Eva Silverstein, Joel Smoller, Steve Strogatz, Leonard Susskind, Ya Soibelman, Erik Swanson, Max Tegmark, Ravi Vakil, Fernando Rodriguez Villegas, Dwight Vincen Dan Waldram, Devin Walker, Brian Wecht, Toby Wiseman, Jeff Wu, Chen Ning Yang, Donald Zeyl and others.

Many of the concepts in this book are difficult to illustrate, and we were fortunate to be able draw on the extraordinary graphic talents of Xiaotian (Tim) Yin and Xianfeng (David) Gu of the Ston Brook Computer Science Department, who were assisted in turn by Huayong Li and Wei Zen Additional help on the graphics front was provided by Andrew Hanson (the premier renderer

Calabi-Yau manifolds), John Oprea, and Richard Palais, among others.

I thank my many friends and relatives, including Will Blanchard, John De Lancey, Ross Eatma Evan Hadingham, Harris McCarter, and John Tibbetts, who read drafts of the book proposal ar chapters or otherwise offered advice and encouragement along the way. Both my coauthor and I a grateful for the invaluable administrative assistance provided by Maureen Armstrong , Lily Chan, Ha Xu, and Gena Bursan.

Several books proved to be valuable references. Among them are *The Elegant Universe* by Bria Greene, *Euclid's Window* by Leonard Mlodinow, *Poetry of the Universe* by Robert Osserman, and *Th Cosmic Landscape* by Leonard Susskind.

The Shape of Inner Space might never have gotten off the ground were it not for the help of Jol Brockman, Katinka Matson, Michael Healey, Max Brockman, Russell Weinberger, and others at th Brockman, Inc., literary agency. T. J. Kelleher of Basic Books had faith in our manuscript when othe did not, and—with the help of his colleague, Whitney Casser—worked hard to get our book into presentable form. Kay Mariea, the project editor at Basic Books, shepherded our manuscript throu its many stages, and Patricia Boyd provided expert copyediting, teaching me that "the same" ar "exactly the same" are exactly the same thing.

Finally, I'm especially grateful for the support from my family members—Melissa, Juliet, ar Pauline, along with my parents Lorraine and Marty, my brother Fred, and my sister Sue—who acted if six-dimensional Calabi-Yau manifolds were the most fascinating thing in the world, not realizir that these manifolds are, in fact, out of this world.

—STEVE NADIS, CAMBRIDGE, MASSACHUSETTS, MARCH 2010

PRELUDE

THE SHAPES OF THINGS TO COME

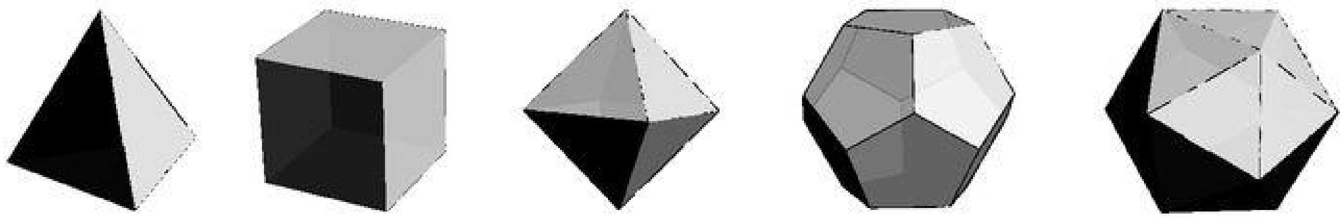
God ever geometrizes.

—PLA

In the year 360 B.C. or thereabouts, Plato published *Timaeus*—a creation story told in the form of dialogue between his mentor, Socrates, and three others: Timaeus, Hermocrates, and Critias. Timaeu likely a fictitious character who is said to have come to Athens from the southern Italian city of Locr is an "expert in astronomy [who] has made it his main business to know the nature of the universe. Through Timaeus, Plato presents his own theory of everything, with geometry playing a central role those ideas.

Plato was particularly fascinated with a group of convex shapes, a special class of polyhedra th have since come to be known as the Platonic solids. The faces of each solid consist of identic polygons. The tetrahedron, for example, has four faces, each a triangle. The hexahedron, or cube, made up of six squares. The octahedron consists of eight triangles, the dodecahedron of twel pentagons, and the icosahedron of twenty triangles.

Plato did not invent the solids that bear his name, and no one knows who did. It is general believed, however, that one of his contemporaries, Theaetetus, was the first to prove that five, an only five, such solids—or *convex regular polyhedra*, as they're called—exist. Euclid gave a comple mathematical description of these geometric forms in *The Elements*.



0.1—The five *Platonic solids*, named for the Greek philosopher Plato: the tetrahedron, hexahedron (cube), octahedron, dodecahedron, and icosahedron. The prefixes derive from the number of face four, six, eight, twelve, and twenty, respectively. One feature of these solids that no other conve polyhedra satisfy is that all their faces, edges, and angles (between two edges) are congruent.

The Platonic solids have several intriguing properties, some of which turn out to be equivalent wa of ~~describing them. For each type of solid, the same number of faces meet at each of the corner~~ point or vertices. One can draw a sphere around the solid that touches every one of those vertices something that's not possible for polyhedra in general. Moreover, the angle of each vertex, where tw edges meet, is always the same. The number of vertices plus faces equals the number of edges pl two.

Plato attached a metaphysical significance to the solids, which is why his name is forever link with them. In fact, the convex regular polyhedra, as detailed in *Timaeus*, formed the very essence his cosmology. In Plato's grand scheme of things, there are four basic elements: earth, air, fire, an water. If we could examine these elements in fine detail, we'd notice that they are composed minuscule versions of the Platonic solids: Earth would thus consist of tiny cubes, the air octahedrons, fire of tetrahedrons, and water of icosahedrons. "One other construction, a fifth, st remained," Plato wrote in *Timaeus*, referring to the dodecahedron. "And this one god used for th whole universe, embroidering figures on it."[2]

As seen today, with the benefit of 2,000-plus years of science, Plato's conjecture looks rath dubious. While there is, at present, no ironclad agreement as to the basic building blocks of th universe—be they leptons and quarks, or hypothetical subquarks called preons, or equally hypothetic and even tinier strings—we do know that it's not just earth, air, fire, and water embroidered upon on giant dodecahedron. Nor do we believe that the properties of the elements are governed strictly by th shapes of Platonic solids.

On the other hand, Plato never claimed to have arrived at the definitive theory of nature. H considered *Timaeus* a "likely account," the best he could come up with at the time, while concedi that others who came after him might very well improve on the picture, perhaps in a dramatic way. A Timaeus states midway into his discourse: "If anyone puts this claim to the test and discovers that isn't so, his be the prize, with our congratulations."[3]

There's no question that Plato got many things wrong, but viewing his thesis in the broadest sens it's clear that he got some things right as well. The eminent philosopher showed perhaps the greate wisdom in acknowledging that what he put forth might not be true, but that another theory, perha building on some of his ideas, could be true. The solids, for instance, are objects of extraordina symmetry: The icosahedron and dodecahedron, for instance, can be rotated sixty ways (which, n coincidentally, turns out to be twice the number of edges in each shape) and still look the same. basing his cosmology on these shapes, Plato correctly surmised that symmetry ought to lie at the hea of any credible description of nature. For if we are ever to produce a real theory of everything— which all the forces are unified and all the constituents obey a handful (or two) of rules—we'll need uncover the underlying symmetry, the simplifying principle from which everything else springs.

It hardly bears mentioning that the symmetry of the solids is a direct consequence of their preci shape or geometry. And this is where Plato made his second big contribution: In addition to realizi that mathematics was the key to fathoming our universe, he introduced an approach we now call th geometrization of physics—the same leap that Einstein made. In an act of great prescience, Pla sug~~gested that the elements of nature, their qualities, and the forces that act upon them may all~~ be th result of some hidden geometrical structure that conducts its business behind the scenes. The world w see, in other words, is a mere reflection of the underlying geometry that we might not see. This is notion dear to my heart, and it relates closely to the mathematical proof for which I am best known- to the extent that I am known at all. Though it may strike some as far-fetched, yet another case geometric grandstanding, there just might be something to this idea, as we’ll see in the pages ahead.

One

A UNIVERSE IN THE MARGINS

The invention of the telescope, and its steady improvement over the years, helped confirm what h become a truism: There's more to the universe than we can see. Indeed, the best available evidenc suggests that nearly three-fourths of all the stuff of the cosmos lies in a mysterious, invisible for called dark energy. Most of the rest—excluding only the 4 percent composed of ordinary matter th includes us—is called dark matter. And true to form, it too has proved "dark" in just about ever respect: hard to see and equally hard to fathom.

The portion of the cosmos we can see forms a sphere with a radius of about 13.7 billion light-year This sphere is sometimes referred to as a Hubble volume, but no one believes that's the full extent the universe. According to the best current data, the universe appears to extend limitlessly, wi straight lines literally stretching from here to eternity in every direction we can point.

There's a chance, however, that the universe is ultimately curved and bounded. But even if it is, th allowable curvature is so slight that, according to some analyses, the Hubble volume we see is just o out of at least one thousand such volumes that must exist. And a recently launched space instrumen the Planck telescope, may reveal within a few years that there are at least *one million* Hubble volum out there in the cosmos, only one of which we'll ever have access to.[1] I'm trusting the astrophysicis on this one, realizing that some may quarrel with the exact numbers cited above. One fact, howeve appears to be unassailable: What we see is just the tip of the iceberg.

At the other extreme, microscopes, particle accelerators, and various imaging devices continue reveal the universe on a miniature scale, illuminating a previously inaccessible world of cell molecules, atoms, and smaller entities. By now, none of this should be all that surprising. We full expect our telescopes to probe ever deeper into space, just as our microscopes and other tools brin more of the invisible to light.

But in the last few decades—owing to developments in theoretical physics, plus some advances geometry that I've been fortunate enough to participate in—there has been another realization that even more startling: Not only is there more to the universe than we can see, but there may even more dimensions, and possibly quite a few more than the three spatial dimensions we're intimate acquainted with.

That's a tough proposition to swallow, because if there's one thing we know about our world— there's one thing our senses have told us from our first conscious moments and first gropin explorations—it's the number of dimensions. And that number is three. Not three, give or take dim~~ension or so, but exactly three. Or so it seemed for the longest time. But maybe, just maybe~~, the are additional dimensions so small that we haven't noticed them yet. And despite their modest siz they could be crucial in ways we could not have possibly appreciated from our entrenched, thre dimensional perspective.

While this may be hard to accept, we've learned in the past century that whenever we stray far fro the realm of everyday experience, our intuition can fail us. If we travel extremely fast, speci relativity tells us that time slows down, not something you're likely to intuit from common sense. we make an object extremely small, according to the dictates of quantum mechanics, we can't sa exactly where it is. When we do experiments to determine whether the object has ended up behi Door A or Door B, we find it's neither here nor there, in the sense that it has no absolute positio (And it sometimes may appear to be in both places at once!) Strange things, in other words, can a will happen, and it's possible that tiny, hidden dimensions are one of them.

If this idea is true, then there might be a kind of universe in the margins—a critical chunk of re estate tucked off to the side, just beyond the reach of our senses. This would be revolutionary in tw ways. The mere existence of extra dimensions—a staple of science fiction for more than a hundre years—would be startling enough on its own, surely ranking among the greatest findings in the histo of physics. But such a discovery would really be a starting point rather than an end unto itself. For ju as a general might obtain a clearer perspective on the battlefield by observing the proceedings from hilltop or tower and thereby gaining the benefit of a vertical dimension, so too may our laws physics become more apparent, and hence more readily discerned, when viewed from a highe dimensional vantage point.

We're familiar with travel in three basic directions: north or south, east or west, and up or down. (O equivalently, left or right, backward or forward, and up or down.) Wherever we go—whether it driving to the grocery store or flying to Tahiti—we move in some combination of those thre independent directions. So familiar are we with these dimensions that trying to conceive of additional dimension—and figuring out exactly where it would point—might seem impossible. For long while, it seemed as if what you see is what you get. In fact, more than two thousand years ag Aristotle argued as much in his treatise *On the Heavens*: "A magnitude if divisible one way is a line, two ways a surface, and if three a body. Beyond these there is no other magnitude, because the thre dimensions are all that there are."[2] In A.D. 150, the astronomer and mathematician Ptolemy tried prove that four dimensions are impossible, insisting that you cannot draw four mutually perpendicul lines. A fourth perpendicular, he contended, would be "entirely without measure and witho definition."[3] His argument, however, was less a rigorous proof than a reflection of our inability bo to visualize and to draw in four dimensions.

To a mathematician, a dimension is a "degree of freedom"—an independent way of moving space. A fly buzzing around over our heads is free to move in any direction the skies permi Assuming there are no obstacles, it has three degrees of freedom. Suppose that fly lands on a parki lot and gets stuck in a patch of fresh tar. While it is temporarily immobilized, the fly has zero degre of freedom and is effectively confined to a single spot—a zero-dimensional world. But this creature ~~persistent and, after some struggle, wrests itself free from the tar, though injuring its wing~~ in th process. Unable to fly, it has two degrees of freedom and can roam the surface of the parking lot will. Sensing a predator—a ravenous frog, perhaps—our hero seeks refuge in a rusted tailpipe lying the lot. The fly thus has one degree of freedom, trapped at least for now in the one-dimensional linear world of this narrow pipe.

But is that all there is? Does a fly buzzing through the air, stuck in tar, crawling on the asphalt, o making its way through a pipe include all the possibilities imaginable? Aristotle or Ptolemy wou have said yes, but while this may be the case for a not terribly enterprising fly, it is not the end of th story for contemporary mathematicians, who typically find no compelling reason to stop at thre dimensions. On the contrary, we believe that to truly understand a concept in geometry, such a curvature or distance, we need to understand it in all possible dimensions, from zero to *n*, where *n* ca be a very big number indeed. Our grasp of that concept will be incomplete if we stop at thre dimensions—the point being that if a rule or law of nature works in a space of any dimension, it more powerful, and seemingly more fundamental, than a statement that only applies in a particul setting.

Even if the problem you're grappling with pertains to just two or three dimensions, you might st secure helpful clues by studying it in a variety of dimensions. Let's return to our example of the f flitting about in three-dimensional space, which has three directions in which to move, or thre degrees of freedom. Yet let's suppose another fly is moving freely in that same space; it too has thre degrees of freedom, and the system as a whole has suddenly gone from three to six dimensions—wi six independent ways of moving. With more flies zigzagging through the space—all moving on the own without regard to the other—the complexity of the system goes up, as does the dimensionality.

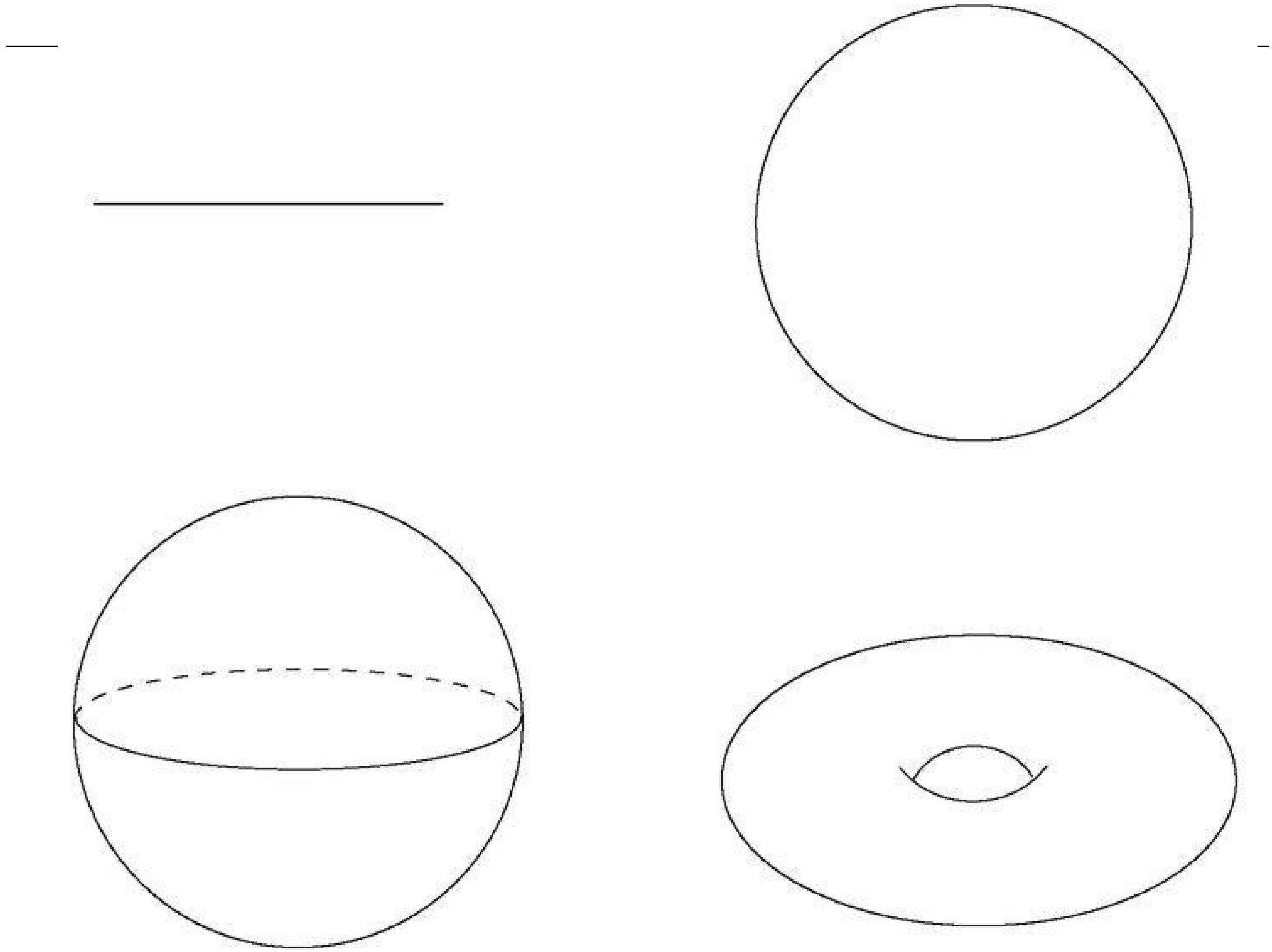
One advantage in looking at higher-dimensional systems is that we can divine patterns that mig be impossible to perceive in a simpler setting. In the next chapter, for instance, we'll discuss the fa that on a spherical planet, hypothetically covered by a giant ocean, all the water cannot flow in th same direction—say, from west to east—at every point. There have to be some spots where the wat is not moving at all. Although this rule applies to a two-dimensional surface, it can only be derived b looking at a much higher-dimensional system in which all possible configurations—all possib movements of tiny bits of water on the surface—are considered. That's why we continually push t higher dimensions to see what it might lead to and what we might learn.

One thing that higher dimensions lead to is greater complexity. In topology, which classifies objec in terms of shape in the most general sense, there are just two kinds of one-dimensional spaces: a lin (a curve with two open ends) and a circle (a closed curve with no ends). There aren't any oth possibilities. Of course, the line could be squiggly, or the closed curve oblong, but those are questio of geometry, not topology. The difference between geometry and topology is like the differenc between looking at the earth's surface with a magnifying glass and going up in a rocket ship an surveying the planet as a whole. The choice comes down to this: Do you insist on knowing every la detail—every ridge, undulation, and crevice in the surface—or will the big picture ("a giant rou ball") suffice? Whereas geometers are often concerned with identifying the exact shape and curvatu of some object, topologists only care about the overall shape. In that sense, topology is a holist discipline, which stands in sharp contrast to other areas of mathematics in which advances a typically made by taking complicated objects and breaking them down into smaller and simpl pieces.

As for how this ties into our discussion of dimensions, there are—as we’ve said—just two bas one-dimensional shapes in topology: A straight line is identical to a wiggly line, and a circle identical to any “loop”—oblong, squiggly, or even square—that you can imagine. The number of tw dimensional spaces is similarly restricted to two basic types: either a sphere or a donut. A topologi considers any two-dimensional surface without holes in it to be a sphere, and this includes everyda geometric shapes such as cubes, prisms, pyramids, and even watermelon-like objects called ellipsoid

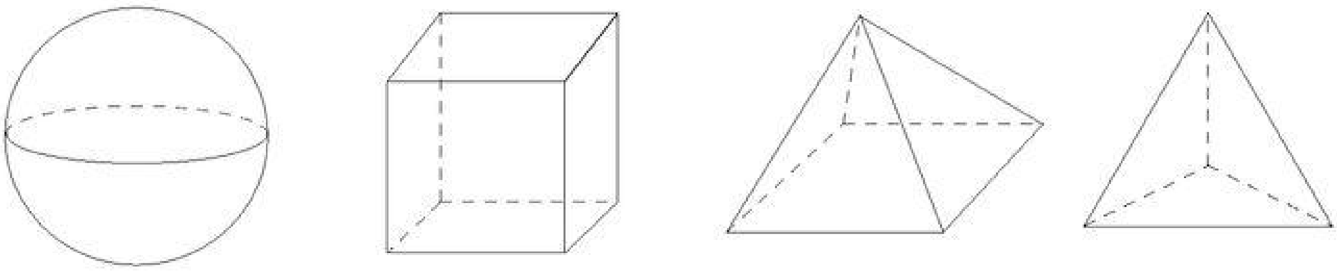
The presence of the hole in the donut or the lack of the hole in the sphere makes all the difference this case: No matter how much you manipulate or deform a sphere—without ripping a hole in it, th is—you’ll never wind up with a donut, and vice versa. In other words, you cannot create new holes an object, or otherwise tear it, without changing its topology. Conversely, topologists regard tw shapes as functionally equivalent if—supposing they are made out of malleable clay or Play-Doh- one shape can be molded into the other by squeezing and stretching but not ripping.

A donut with one hole is technically called a *torus*, but a donut-like surface could have any numb of holes. Two-dimensional surfaces that are both compact (closed up and finite in extent) an orientable (double-sided) can be classified by the number of holes they have, which is also known their *genus*. Objects that look quite different in two dimensions are considered topologically identic if they have the same genus.

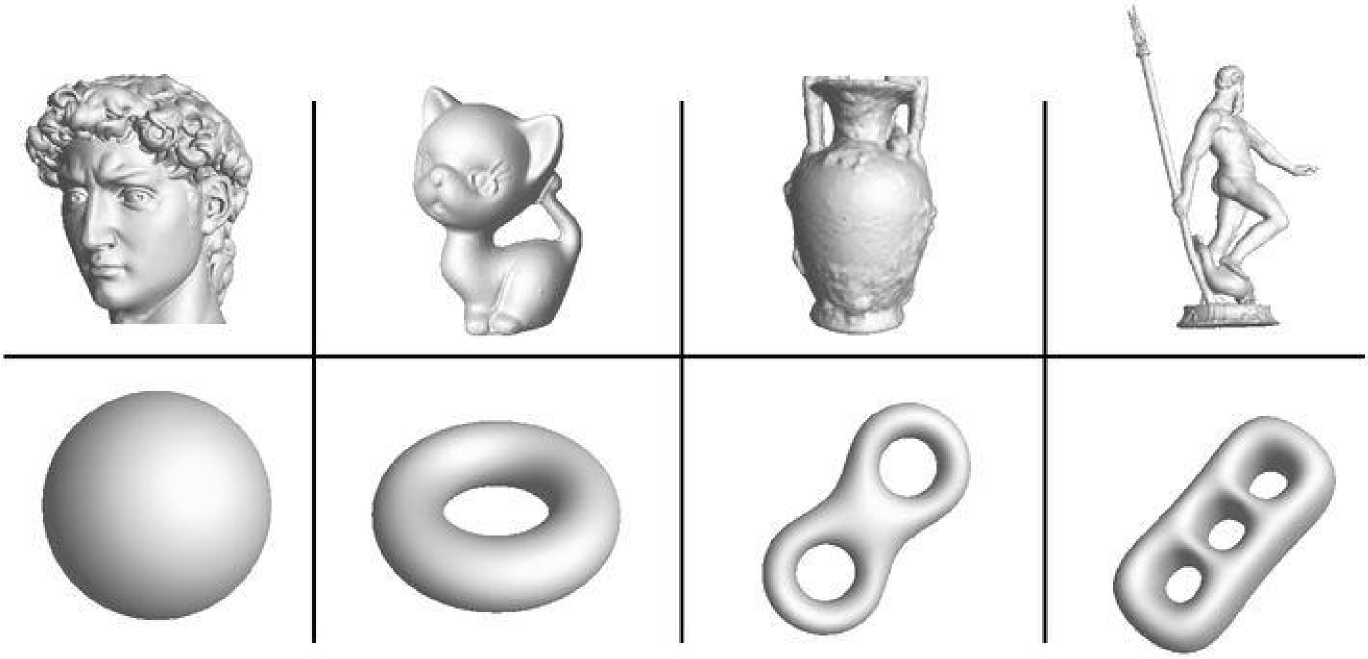


1.1—In topology, there are just two kinds of one-dimensional spaces that are fundamentally distin
from each other: a line and a circle. You can make a circle into all kinds of loops, but you can't turn
into a line without cutting it.

Two-dimensional surfaces, which are *orientable*—meaning they have two sides like a beach ba
rather than just one side like a Möbius strip—can be classified by their *genus*, which can be thoug
of, in simple terms, as the number of holes. A sphere of genus 0, which has no holes, is therefo
fundamentally distinct from a donut of genus 1, which has one hole. As with the circle and line, yo
can't transform a sphere into a donut without cutting a hole through the middle of it.



1.2—In topology, a sphere, cube, and tetrahedron—among other shapes—are all considered equivale
because each can be fashioned from the other by bending, stretching, or pushing, without their havi
to be torn or cut.



1.3—Surfaces of genus 0, 1, 2, and 3; the term *genus* refers to the number of holes.

The point made earlier about there being just two possible two-dimensional shapes—a donut or sphere—is only true if we restrict ourselves to orientable surfaces, and those are the surfaces we' generally be referring to in this book. A beach ball, for example, has two sides, an inside and a outside, and the same goes for a tire's innertube. There are, however, more complicated cases- single-sided, "nonorientable" surfaces such as the Klein bottle and Möbius strip—where the foregoin is not true.

In dimensions three and beyond, the number of possible shapes widens dramatically. I contemplating higher-dimensional spaces, we must allow for movements in directions we can readily imagine. We're not talking about heading somewhere in between north and west like northwe or even "North by Northwest." We're talking about heading off the grid altogether, following arrow in a coordinate system that has yet to be drawn.

One of the first big breakthroughs in charting higher-dimensional space came courtesy of Ren Descartes, the seventeenth-century French mathematician, philosopher, scientist, and writer, thoug his work in geometry stands foremost for me. Among other contributions, Descartes taught us th thinking in terms of coordinates rather than pictures can be extremely productive. The labeling syste he invented, which is now called the Cartesian coordinate system, united algebra and geometry. In narrow sense, Descartes showed that by drawing *x*, *y*, and *z* axes that intersect in a point and are a perpendicular to each other, one can pin down any spot in three-dimensional space with just thre numbers—the *x*, *y*, and *z* coordinates. But his contribution was much broader than that, as he vast enlarged the scope of geometry and did so in one brilliant stroke. For with his coordinate system hand, it became possible to use algebraic equations to describe complex, higher-dimension geometric figures that are not readily visualized.

Using this approach, you can think about any dimension you want—not just (x, y, z) but (a, b, c, e, f)~~ or (j, k, l, m, n, o, p, q, r, s)—the dimension of a given space being the number of coordinates~~ yo need to determine the location of a given point. Armed with this system, one could contempla higher-dimensional spaces of any order—and do various calculations concerning them—witho having to worry about trying to draw them.

The great German mathematician Georg Friedrich Bernhard Riemann took off with this idea tw centuries later and carried it far. In the 1850s, while working on the geometry of curved (non Euclidean) spaces—a subject that will be taken up in the next chapter—Riemann realized that the spaces were not restricted in terms of the number of dimensions. He showed how distance, curvatur and other properties in such spaces could be precisely computed. And in an 1854 inaugural lecture which he presented principles that have since come to be known as Riemannian geometry, speculated on the dimensionality and geometry of the universe itself. While still in his twentie Riemann also began work on a mathematical theory that attempted to tie together electricit magnetism, light, and gravity—thereby anticipating a task that continues to occupy scientists to th day.

Although Riemann helped free up space from the limitations of Euclidean flatness and thre dimensions, physicists did not do much with that idea for decades. Their lack of interest may hav stemmed from the absence of experimental evidence to suggest that space was curved or that an dimensions beyond three existed. What it came down to was that Riemann's advanced mathemati had simply outpaced the physics of his era, and it would take time—another fifty years or so—for th physicists, or at least one physicist in particular, to catch up. The one who did was Albert Einstein.

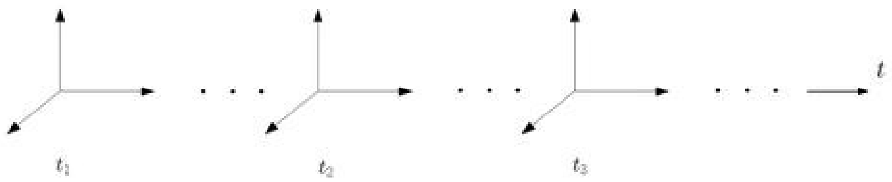
In developing his special theory of relativity—which was first presented in 1905 and furth advanced in the years after, culminating in the general theory of relativity—Einstein drew on an ide that was also being explored by the German mathematician Hermann Minkowski, namely, that time inextricably intertwined with the three dimensions of space, forming a new geometrical constru known as *spacetime*. In an unexpected turn, time itself came to be seen as the fourth dimension th Riemann had incorporated decades before in his elegant equations.

Curiously, the British writer H. G. Wells had anticipated this same outcome ten years earlier in h novel *The Time Machine*. As explained by the Time Traveller, the main character of that book, "ther are really four dimensions, three which we call the three planes of Space, and a fourth, Time. There i however, a tendency to draw an unreal distinction between the former three dimensions and th latter."[4]

Minkowski said pretty much the same thing in a 1908 speech—except that in this case, he had th mathematics to back up such an outrageous claim: "Henceforth space by itself, and time by itself, a doomed to fade away into mere shadows, and only a kind of union of the two will preserve a independent reality."[5] The rationale behind the marriage of these two concepts—if, indeed, marriag ever have a rationale—is that an object moves not only through space but through time as well. It thu takes four coordinates, three of space and one of time, to describe an event in four-dimension spacetime (x, y, z, t).

Although the idea may seem slightly intimidating, it can be expressed in extremely mundane term Suppose you make plans to meet somebody at a shopping mall. You note the location of the buildir —say it's at the corner of First Street and Second Avenue—and decide to meet on the third floor. Tha takes care of your x, y, and z coordinates. Now all that remains is to fix the fourth coordinate and sett on the time. With those four pieces of information specified, your assignation is all set, barring ar unforeseen circumstances that might intervene. But if you want to put it in Einstein's terms, yo shouldn't look at it as setting the exact place for this little get-together, while separately agreeing o the time. What you're really spelling out is the location of this event in spacetime itself.

So in a single bound, early in the twentieth century, our conception of space grew from the coz three-dimensional nook that had nurtured humankind since antiquity to the more esoteric realm four-dimensional spacetime. This conception of spacetime formed the bedrock on which Einstein theory of gravity, the general theory of relativity, was soon built. But is that the end of the line, as w asked once before? Does the buck stop there, at four dimensions, or can our notion of spacetime gro further still? In 1919, a possible answer to that question arrived unexpectedly in the form of manuscript sent to Einstein for review by a then-unknown German mathematician, Theodor Kaluza.



1.4—As we don't know how to draw a picture in four dimensions, this is a rather crude, conceptu rendering of four-dimensional *spacetime*. The basic idea of spacetime is that the three spati dimensions of our world (represented here by the x-y-z coordinate axis) have essentially the san status as a fourth dimension—that being time. We think of time as a *continuous variable* that's alway changing, and the figure shows snapshots of the coordinate axis at various moments frozen in time: t t_2, t_3, and so forth. In this way, we're trying to show that there are four dimensions overall: three space plus the additional one labeled by time.

In Einstein's theory, it takes ten numbers—or ten *fields*—to precisely describe the workings gravity in four dimensions. The force can be represented most succinctly by taking those ten numbe and arranging them in a four-by-four matrix more formally known as a *metric tensor*—a square tab of numbers that serves as a higher-dimensional analogue of a ruler. In this case, the metric has sixtee entries in all, only ten of which are independent. Six of the numbers repeat because gravity, along wi the other fundamental forces, is inherently symmetrical.

In his paper, Kaluza had basically taken Einstein's general theory of relativity and added an extr dimension to it by expanding the four-by-four matrix to a five-by-five one. By expanding spacetime the fifth dimension, Kaluza was able to take the two forces known at the time, gravity ar electromagnetism, and combine them into a single, unified force. To an observer in the fiv dimensional world that Kaluza envisioned, those forces would be one and the same, which is what w ~~mean by unification. But in a four-dimensional world, the two can’t go together; they would ap~~pear be wholly autonomous. You could say that’s the case simply because both forces do not fit into th same four-by-four matrix. The additional dimension, however, provides enough extra elbow room fo both of them to occupy the same matrix and hence be part of the same, more all-encompassing force.

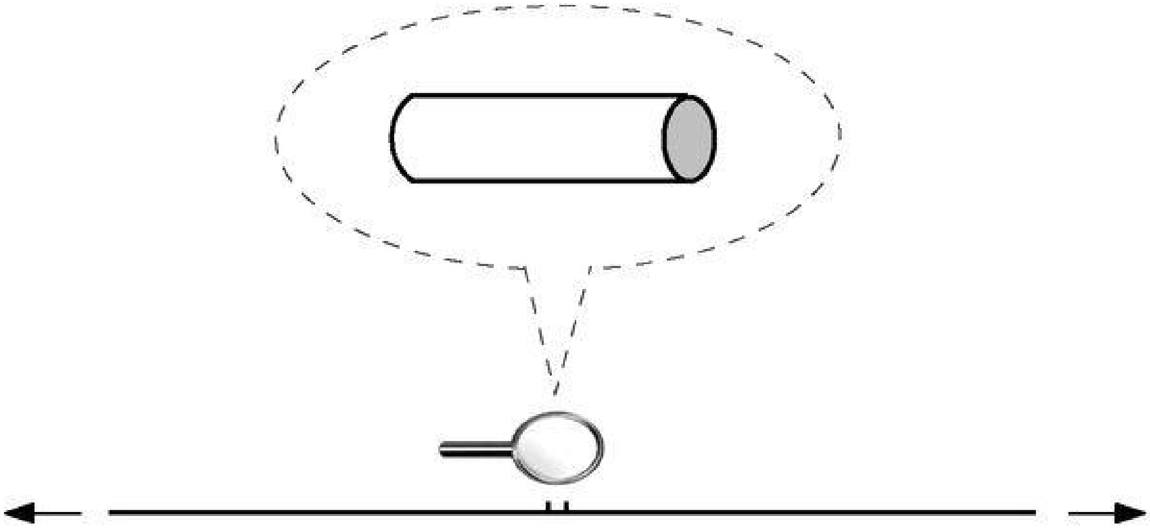
I may get in trouble for saying this, but I believe that only a mathematician would have been bo enough to think that higher-dimensional space would afford us special insight into phenomena th we’ve so far only managed to observe in a lower-dimensional setting. I say that becau mathematicians deal with extra dimensions all the time. We’re so comfortable with that notion, w don’t give it a moment’s thought. We could probably manipulate extra dimensions in our slee without interfering with the REM phase.

Even if I think that only a mathematician would have made such a leap, in this case, remarkably, was a mathematician building on the work of a physicist, Einstein. (And another physicist, Osk Klein, whom we’ll be discussing momentarily, soon built on that mathematician’s work.) That’s why like to position myself at the interface between these two fields, math and physics, where a lot interesting cross-pollination occurs. I’ve hovered around that fertile zone since the 1970s and hav managed to get wind of many intriguing developments as a result.

But returning to Kaluza’s provocative idea, people at the time were puzzled by a question that equally valid today. And it’s one that Kaluza undoubtedly grappled with as well: If there really is fifth dimension—an entirely new direction to move at every point in our familiar four-dimension world—how come nobody has seen it?

The obvious explanation is that this dimension is awfully small. But where would it be? One way get a sense of that is to imagine our four-dimensional universe as a single line that extends endless in both directions. The idea here is that the three spatial dimensions are either extremely big infinitely large. We’ll also assume that time, too, can be mapped onto an infinite line—an assumptio that may be questionable. At any rate, each point w on what we’ve thought of as a line actual represents a particular point (x, y, z, t) in four-dimensional spacetime.

In geometry, lines are normally just length, having no breadth whatsoever. But we’re going to allo for the possibility that this line, when looked at with an exceedingly powerful magnifying glas actually has some thickness. When seen in this light, our line is not really a line at all but rather a extremely slender cylinder or “garden hose,” to choose the standard metaphor. Now, if we slice ou hose at each point w, the cross-section of that cut will be a tiny circle, which, as we’ve said, is a on dimensional curve. The circle thus represents the extra, fifth dimension that is “attached,” in a sens to every single point in four-dimensional spacetime.



1.5—Let's picture our infinite, four-dimensional spacetime as a line that extends endlessly in bo directions. A line, by definition, has no thickness. But if we were to look at that line with a magnifyi glass, as suggested in the Kaluza-Klein approach, we might discover that the line has some thickne after all—that it is, in fact, harboring an extra dimension whose size is set by the diameter of t circle hidden within.

A dimension with that characteristic—being curled up in a tiny circle—is technically referred to being compact. The word *compact* has a fairly intuitive meaning: Physicists sometimes say that compact object or space is something you could fit into the trunk of your car. But there's a mo precise meaning as well: If you travel in one direction long enough, it is possible to return to the sa spot. Kaluza's five-dimensional spacetime includes both extended (infinite) and compact (finit dimensions.

But if that picture were correct, why wouldn't we notice ourselves moving around in circles in th fifth dimension? The answer to that question came in 1926 from Oskar Klein, the Swedish physic who carried Kaluza's idea a step further. Drawing on quantum theory, Klein actually calculated th size of the compact dimension, arriving at a number that was tiny indeed—close to the so-call Planck length, which is about as small as you can get—around 10^{-30} cm in circumference.[6] And that how a fifth dimension could exist, yet remain forever unobservable. There is no foreseeable means which we could see this minuscule dimension; nor could we detect movements within it.

Kaluza-Klein theory, as the work is now known, was truly remarkable, showing the potential extra dimensions to demystify the secrets of nature. After sitting on Kaluza's original paper for mo than two years, Einstein wrote back saying he liked the idea "enormously."[7] In fact, he liked the id enough to pursue Kaluza-Klein-inspired approaches (sometimes in collaboration with the physic Peter Bergmann) off and on over the next twenty years.

But ultimately, Kaluza-Klein theory was discarded. In part this was because it predicted a partic that has never been shown to exist, and in part because attempts to use the theory to compute the rat of an electron's mass to its charge went badly awry. Furthermore, Kaluza and Klein—as well

Einstein after them—were trying to unify only electromagnetism and gravity, as they didn't kno
~~about the weak and strong forces, which were not well understood until the latter half of the twentie~~
century. So their efforts to unify all the forces were doomed to failure because the deck they we
playing with was still missing a couple of important cards. But perhaps the biggest reason that Kaluz
Klein theory was cast aside had to do with timing: It was introduced just as the quantum revolutio
was beginning to take hold.

Whereas Kaluza and Klein put geometry at the center of their physical model, quantum theory not only an ungeometric approach, but also one that directly conflicts with conventional geomet (which is the subject of Chapter 14). In the wake of the upheaval that ensued as quantum theory swe over physics in the twentieth century, and the amazingly productive period that followed, it too almost fifty years for the idea of new dimensions to be taken seriously again.

General relativity, the geometry-based theory that encapsulates our current understanding of gravit has also held up extraordinarily well since Einstein introduced it in 1915, passing every experiment test it has faced. And quantum theory beautifully describes three of the known forces: th electromagnetic, weak, and strong. Indeed, it is the most precise theory we have, and "probably th most accurately tested theory in the history of human thought," as Harvard physicist Andre Strominger has claimed.[8] Predictions of the behavior of an electron in the presence of an electric fiel for example, agree with measurements to ten decimal points.

Unfortunately, these two very robust theories are totally incompatible. If you try to mix gener relativity with quantum mechanics, the combination can create a horrific mess. The trouble aris from the quantum world, where things are always moving or fluctuating: The smaller the scale, th bigger those fluctuations get. The result is that on the tiniest scales, the turbulent, ever-changin picture afforded by quantum mechanics is totally at odds with the smooth geometric picture spacetime upon which the general theory of relativity rests.

Everything in quantum mechanics is based on probabilities, and when general relativity is throw into the quantum model, calculations often lead to infinite probabilities. When infinities pop up as matter of course, that's a tipoff that something is amiss in your calculations. It's hardly an ideal sta of affairs when your two most successful theories—one describing large objects such as planets an galaxies, and the other describing tiny objects such as electrons and quarks—combine to give yo gibberish. Keeping them separate is not a satisfactory solution, either, because there are places, suc as black holes, where the very large and very small converge, and neither theory on its own can mak sense of them. "There shouldn't be laws of physics," Strominger maintains. "There should be just on law and it ought to be the nicest law around."[9]

Such a sentiment—that the universe can and should be describable by a "unified field theory" th weaves all the forces of nature into a seamless whole—is both aesthetically appealing and tied to th notion that our universe started with an intensely hot Big Bang. At that time, all the forces would hav been at the same unimaginably high energy level and would therefore act as if they were a sing