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The Companion Guide to The Mathematical Experience

Study Edition

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Part I

Introduction to this Companion Guide

A Note to Instructors

The first *Mathematical Experience* appeared in 1981. At that time, only a few years ago, it was commonly believed that it was impossible to make contemporary mathematics meaningful to the intelligent non-mathematician. Since then, dozens of popular books on contemporary mathematics have been published. James Gleick's *Chaos* was a long-run best seller. John Casti is producing a continuing series of such books.

In technology and invention, it's a commonplace that knowing what's possible is the most important ingredient of successful innovation. Perhaps the first *Mathematical Experience* changed people's idea about what's possible in exposition of advanced contemporary mathematics.

Alert readers recognized the book as a work of philosophy—a humanist philosophy of mathematics. It was far out, "maverick" (Philip Kitcher's term), virtually out of contact with official academic philosophy of mathematics. In the past 15 years, humanist philosophy of mathematics has bloomed. There are anthologies, symposia, a journal. The far-out maverick of 15 years ago might be the mainstream in a few years.

The first *Mathematical Experience* was a trade book, not a textbook. It was sold in book stores, not in professor's offices. But we heard over and over of college teachers using it, in the United States, Europe, Australia, Hong Kong, Israel. It's used in two different ways: "Math for liberal arts students" in colleges of art and science, and courses for future teachers, especially secondary math teachers, in colleges of education.

In mathematics teaching, it's a commonplace that "Mathematics isn't a spectator sport." You learn by doing, especially doing problems. Like all truisms, this is half true. Mathematics education as doing, doing, doing—no thinking, no conversation, no contemplation—can seem dreary. An artist isn't prohibited from occasional art appreciation—quite the contrary. You can't learn practical skill as a spectator, but you can learn good taste, among other things.

The first edition invited the reader to appreciate mathematics, contemplate it, participate in a conversation about it. It contained no problems. If a teacher selected it, he/she had to supply what the book lacked. The study edition will be more convenient for both teacher and student. It aims for balance between doing and thinking. There are plenty of problems, generous discussion guides, essay topics, and bibliographies. We've also introduced "projects": connected sequences of problems, rising in difficulty from very easy to a little less easy. They provide extra problem-solving enjoyment, and they make points about the nature of mathematics. We've written a section on differential and integral calculus—a complete course in 15 pages—and a section on the fascinating topic of complex numbers—fascinating from both mathematical and philosophical viewpoints.

The *Standards* of the National Council of Teachers of Mathematics appeared after the first *Mathematical Experience*. To a large extent, they validated our enterprise. We were following the *Standards* before they were written. The study edition does so even more than the first.

No longer are "critical thinking" and "problem solving" only features of mathematics. They've become catchwords in American classrooms. The second *Mathematical Experience* is a part of the dominant trend in American education.

Special Features

In the Text

Topics to Explore are listed in each chapter of the text. In this *Companion Guide* you will find you will find Suggested Topics for Classroom Discussions, Projects, and Tutorials that relate to the **Topics to Explore**. **Suggested Readings** in the text are resources for these topics, and additional video resources for some topics are listed in this *Companion Guide*.

Essay Assignments can also be used as Topics for Classroom Discussion. **Suggested Readings** in the text are resources for these assignments.

In writing essays students can come to understand the extent of their mathematical knowledge. This activity also familiarizes them with the language of mathematics—improving dialogue in the classroom.

Problems can also be used in the classroom as group activities (see this *Companion Guide*, Part iv, *Sample Group Activities*). Group activities encourage students to assume a more active role in the classroom, helping them to see themselves and their classmates (rather than only the instructor) as resources for learning. **Suggested Readings** in the text are resources for these assignments.

Computer Problems: In a course such as we are laying out, computer problems can serve to emphasize a number of important points. Among them are: (1) the verification of mathematical statements and identities; (2) the discovery of new facts through computer experimentation and induction (*not* mathematical induction); (3) the great successes and occasional pitfalls of the computation process; (4) the limitations imposed by the digital language as opposed to the richer existential language of "full" mathematics; (5) the extent to which one needs mathematical knowledge and expertise beyond what is built into commercial mathematical software; (6) the appreciation of the computational infrastructure of our civilization, an infrastructure that is often hidden from view.

In the Companion Guide

Expository Research Papers: Suggestions are given (see Part VI, *Suggestions for Expository Research Papers*) for expository research paper assignments. However, students should also be encouraged to choose their own topics. It is important to provide specific tasks for the students regarding the writing of a research paper and to give them explicit deadlines. These tasks include writing a one-page essay describing the topic they select and what they expect to learn and demonstrate in their paper; submitting a bibliography for the paper; writing an outline; submitting a first draft (see Part III, *Sample Syllabus*).

Topics for Classroom Discussion are useful for general class discussion or as a source of group activities. **Suggested Readings** in the text are resources for these discussions and additional video resources for some topics are listed in this *Companion Guide*. The emphasis for these topics is on *discussion* more than *lecture*. Relinquishing the lecture as the primary mode of instruction, taking the opportunity to hear the student in the classroom, helps to foster an environment in which instructor and students converse to form a community of learners. Conducting class meetings as open forums to discover student interest in selected topics, to motivate and explore objectives for topics, and to discuss possible directions that these topics suggest, helps students recognize their responsibility in the learning process.

Projects are connected sequences of explorations that start with very familiar, "easy" material and gradually lead the student to new discoveries and a glimpse at broader vistas. A project can be assigned as individual homework, worth perhaps two weeks each. Better, it can be carried out in small groups which periodically report to the class as a whole to compare notes. This reporting and comparing leads to work on the project by the class as a whole, with occasional hints and suggestions by the instructor. Such a class session will re-energize the individual students or groups to go further.

Projects can be shortened by omitting the last parts, or enlarged by instructor or students coming up with new directions to pursue. These projects are also models to help the instructor make up projects in line with her background and interests.

Tutorials: Two of the additions to this Companion Guide have been labelled "Tutorials." One is about differential and integral calculus, and the other is about complex numbers. These two topics are essential in any survey of mathematics. We suggest that the instructor provide the class with photocopies of this material.

Each of these sections could have been part of the original text. They are more text-bookish than the rest of the text, since they are straightforward presentations of classical mathematics. But we still strive for a respectable level of literary style, and take every opportunity to tell about historical background and philosophical controversies. The tutorial on calculus is a *tour de force.* In only fifteen pages it explains the guts of the usual one-semester course, including applications and problems .

Pictures and diagrams are essential when teaching calculus. Our artwork was done by Caroline Smith of the University of New Mexico, to whom we are most grateful.

Part II

Chapter Guidelines

Chapter 1

The Mathematical Landscape

What is Mathematics? Where is Mathematics? The Mathematical Community. Tools of the Trade. How Much Mathematics is Now Known? Ulam's Dilemma. How Much Mathematics Can There Be?

Topics for Classroom Discussion

1. An alien has landed. She asks you what mathematics is. How do you answer? What do mathematicians do? Does mathematics change?

2. What is Ulam's dilemma? If you were Ulam, and a reporter for *Newsweek* was interviewing you regarding this dilemma, how would you explain it? Is there anything that can be done about it? Will the *Information Superhighway* help?

3. Can mathematics establish truth? Plato thought so. Eric Temple Bell thought not. What do you think? How are proof and truth related? See, for example, "The Concept of Mathematical Truth" by Gian-Carlo Rota in *Essays in Humanistic Mathematics* (Washington, D.C., Mathematical Association of America, 1993).

4. Some mathematicians have described the process of mathematical research as a kind of "playing around." Discuss.

5. π is the ratio of a circle's circumference to its diameter. It cannot be constructed with a straightedge and a compass. π is irrational. It is a transcendental number. π shows up in number theory, in geometry, in probability. Was π invented or discovered by mathematicians? See, for example, "π and e" by E. C. Titchmarsh in *Mathematics: People, Problems, Results* edited by D. Campbell and J. Higgens (Belmont, CA: Wordsworth International, 1984).

6. Investigate the number *e*. What kind of number is it? Where do we find it in mathematics? Was it invented or discovered by mathematicians? A good reference is *e – The Story of a Number*, by Eli Maor (Princeton: Princeton University Press, 1994).

7. Find reasons for thinking that any brief definition of mathematics must be inadequate.

8. Defend the view that computer science is part of mathematics.

9. Could there be such a thing as "unconscious mathematics" which need not be symbolized in any way, but which leads to certain consequences?

10. How much music is there? How much literature is there? In these instances, how could you make an acceptable definition of "how much"? How could you go about implementing your definition? What could you do with an answer?

Chapter 2
Varieties of Mathematical Experience

The Ideal Mathematician. The Individual and the Culture. The Current Individual and Collective Consciousness. A Physicist Looks at Mathematics. I. R. Shafarevitch and the New Neoplatonism. Unorthodoxies

I. Discussion Topic: Proof

1. In "The Ideal Mathematician" the student asks the ideal mathematician about proof. What is *your* conception of proof? How would you answer the following questions:

a. What is the role of proof in mathematics?

b. Why do mathematicians prove theorems?

c. What does it mean for a mathematical statement to be considered true?

2. Christian Goldbach (1690–1764) conjectured that every even number greater than two is the sum of two odd primes. Do you believe this? Why? Pick ten even numbers and see if they each are the sum of two odd primes.

3. What role do intuition and evidence play in proof? What is the difference between conjecture and a proof?

4. Kurt Gödel (1906–1978) showed that not every true statement is provable in mathematics. Is every provable statement true in mathematics?

II. Discussion Topic: The Many Roads to Proof

1. Is there only one type of proof? If someone asked you to describe what a proof is and what it does, what would you say?

2. The MATHEMATICS! videotape "The Theorem of Pythagoras" (Pasadena: California Institute of Technology, 1988) demonstrates several animated dissection proofs of the Pythagorean theorem. Students can be challenged to construct their own dissection proof or one they see in the videotape.

A nice follow-up to seeing this video is the geometric paradox one encounters in dissecting an 8-inch square into a 5×13-inch rectangle (see *Fibonacci Numbers* by N. N. Vorob'ev). It's very effective in demonstrating the difference between evidence and proof. Fibonacci numbers can be introduced here or later to revisit the paradox. (See page 35 of this *Companion Guide*.)

3. Do you think there is proof in the following professional areas: medicine, physics, law, religion?

4. How would you *prove* that $123 \times 587 = 72201$?

III. Tutorial: Calculus

This is a concise presentation of the main ideas of calculus. Because the topic is so important, we make some unavoidable concessions to the standard textbook format.

Calculus is the heart of modern mathematics, since Newton. It's the part of mathematics most important in science and technology, the part engineers must know.

It grows out of two main problems which at first seem unrelated.

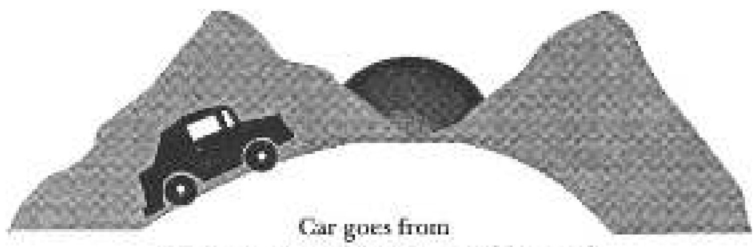
The central discovery of calculus is that these problems are opposites or inverses.

The first main problem in calculus is speed. How fast is something changing? The solution of this problem is "the differential calculus." The second main problem is area. How big is the inside of some curved region? The solution of this problem is "the integral calculus."

First we'll talk about speed. It's easy to find the speed if it's constant. Just divide the distance traveled by the time elapsed. Speed = Distance/Time.

Motion at Constant Speed

But in real motion, speed isn't constant. You start your car at speed zero. You go faster till you get to the speed limit.

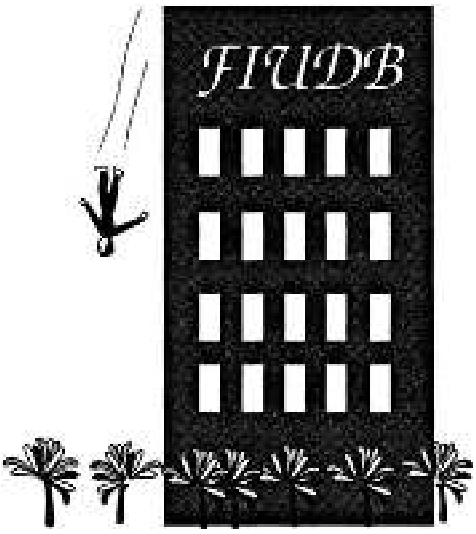


Car goes from
Albuquerque to Santa Fe, variable speed

Motion at Variable Speed

Then to stop, you slow back down to zero. Your speed changes from instant to instant. What is your speed at some particular instant?

Here's another practical example. Nick, our math professor, falls off the First International Unpaid Debts Building in Miami. How fast is he falling? In school we learned that in a vacuum, under the acceleration of gravity, a body falls $16t^2$ feet in t seconds. How fast is he falling after 2 seconds, ignoring air resistance?



Nick falling off the F.I.U.D. Building.

In the time interval between 2 seconds and 2.1 seconds—time lapse of 0.1 second—the distance Nick falls is the difference between how far he had fallen after 2 seconds and how far he had fallen after 2.1 seconds.

$$16(2.1)^2 - 16(2^2) \text{ feet} = 6.56 \text{ feet}.$$

Dividing distance by time (0.1 second), his average speed over that tenth of a second is 65.6 ft/sec. That's the constant speed that would carry him the same distance *in the same time as did his actual fall at the accelerated speed.*

Exercise. Repeat the calculation with a time lapse of 0.01 second. (You'll get an average speed of 64.16 ft/sec, between time 2 seconds and time 2.01 seconds.)

Do it still again, with a *tiny* time lapse, 0.001 seconds. (Nick's average velocity over this time period is 64.01 ft/sec.)

We don't want an *average* speed. We want the *exact* speed at time 2! That means a time lapse of *zero*. The formula Speed = Distance/Time breaks down, because division by zero is meaningless. However, *without setting the time lapse equal to* 0 you've crept closer and closer to 0. You used lapses of 0.1, 0.01, 0.001, and found speeds of 65.6, 64.16, 64.016.

NOW!! A giant conceptual leap! If the average speeds approach a limit as the time lapse approaches zero, we declare, as a *definition*, that this limit *is* the instantaneous speed!! In this example, the limit is 64 ft/sec when $t = 2$. It makes sense! It works! That's what we mean by instantaneous rate of speed or velocity.

NEWTON

LEIBNIZ

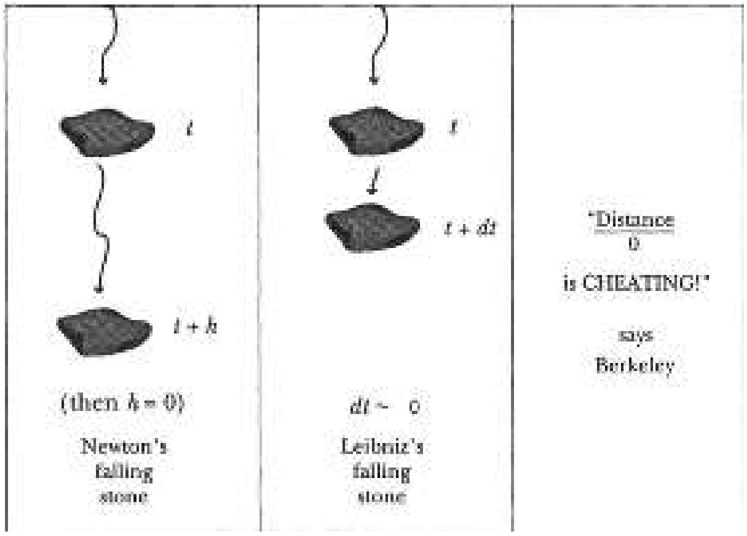
BERKELEY

ROBINSON

This notion of rate as a *limit* took hundreds of years to formulate. Medieval and Renaissance mathematicians calculated a few rates of change without defining mathematically what they wanted. The founders of calculus, Isaac Newton and Gottfried Leibniz, enjoyed a bitter quarrel about priority in the discovery.

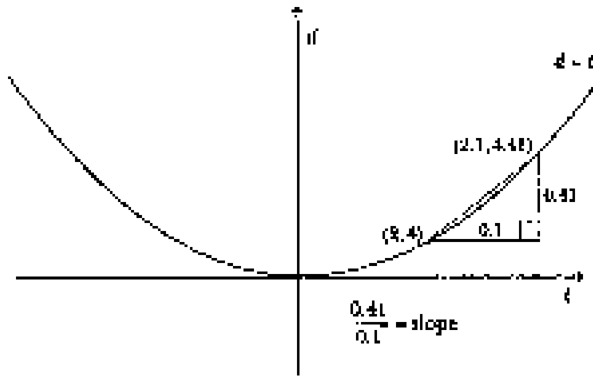
Leibniz' explanation of differentiation was not quite the same as Newton's. Leibniz used an *infinitesimal* increment — a number bigger than zero, yet smaller than any ordinary number.

Then George Berkeley, an empiricist philosopher and Anglican bishop, showed that the reasoning of both Newton and Leibniz was illegitimate. The small increment is sometimes defined to be not zero, sometimes to be zero. This is a contradiction! It was hundreds of years before an answer to Berkeley was found. But meanwhile mathematicians went on with the calculus anyway.



For centuries, people doubted whether infinitesimals make sense. In the 1960s an American logician, Abraham Robinson, used methods from modern logic to make the infinitesimal respectable.

Exercise. Make a graph of this falling body function: distance = time squared or $d = t^2$. (We dropped the 16 to simplify your graphing and our calculating.) This is a quadratic function. Its graph is a parabola. You've studied parabolas, but this calculation is different. Mark the points (2, 4) and (2.1, 4.41) on the parabola. The second is above and right of the first. Draw a straight line (called the "secant") between the two. What's the slope of this line? ("Rise over run.") Rise = $2.1^2 - 2^2 = 0.41$. Run = $2.1 - 2 = 0.1$. Slope = $0.41/0.1 = 4.1$, *which we just found is also the average velocity* (allowing for the factor of 16 which we took out). The average rate of change of distance as a function of time is identical to the slope of its graph! Again, replace 0.1 by 0.01 and 0.001. The corresponding marks on the graph are creeping closer and closer to (2, 4). The slopes of the secants are exactly the numbers you found to approximate the instantaneous rate of change. "In the limit," as the two points approach closer and closer, and the denominator approaches zero, the secant becomes a tangent and its slope becomes the instantaneous speed, called the *derivative*.



Differentiating x^2 is the same as finding its shape. (This graph is called a parabola.)

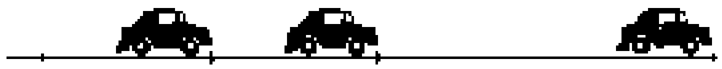
The process of calculating the derivative (the speed) is called differentiation. Simple functions usually have simple derivatives. The derivative of t^n is nt^{n-1}. (n is any number, integer or fraction, positive or negative.) The derivative of the natural logarithm of t is $1/t$. The derivative of e^{at} is ae^{at}. The derivative of $\sin t$ is $\cos t$; of $\cos t$, $-\sin t$. These formulas are always derived in first semester calculus.

> *Exercise.* In a way similar to how you found the rate of change of $f(t) = t^2$ at $t = 2$, find the rate of change of that function at an arbitrary time t. Do the same for the cubic $f(t) = t^3$. Check your answer with the formula in the previous paragraph for t^n.

Now you're ready for the second main problem of calculus, integration—finding the area inside a curve. To solve it, strangely enough, we'll talk about another, quite different-sounding problem. Given the velocity of a moving body—say a car driving down the highway—can we calculate the total distance traveled, at any instant of the trip? This is the opposite of the problem we analyzed above. There we were given the distance and found the velocity.

Start with the simplest case—constant velocity. Suppose that from 2 PM to 3 PM you're driving at a steady 50 miles per hour. How far do you go in that hour? In half an hour? At any time t between 2 PM and 3 PM?

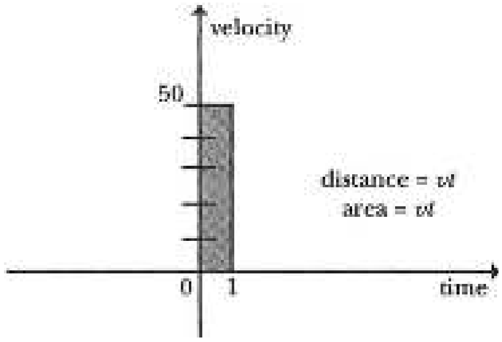
Of course you can answer this—in one hour, 50 miles. In half an hour, 25 miles. In t hours, $50t$ miles—where t can be a fraction.



Driving 50 miles at a constant speed of 50 m.p.h.

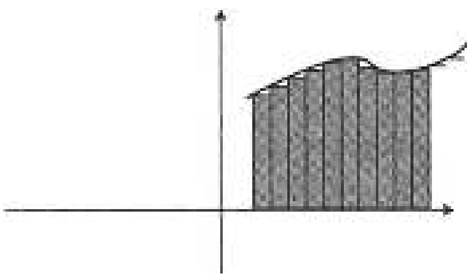
The graph of these facts is simple. Time elapsed is measured on the horizontal time axis. The graph of the constant velocity 50 is a horizontal line 50 units above the time axis. We compute distance by multiplying speed times time—that is, height of the velocity line times length of the time axis from start to finish. The product of these horizontal and vertical lengths equals the area of the rectangle they enclose. Distance is represented graphically as area!

The real problem comes when you vary the speed of your car. Then the graph of $v(t)$, velocity as a function of time, is



On a time-velocity graph, distance = area.

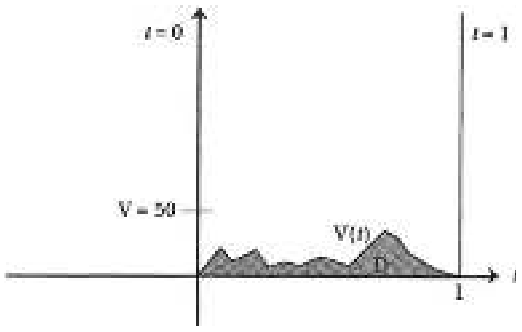
a curve, not a horizontal line. How can we find the distance traveled now? Since we know how to do it in the case of constant speed (horizontal graph), *replace the curved graph by a piecewise horizontal graph.*



Variable velocity is approximated by piecewise-constant velocity.

In other words, instead of a speed varying smoothly, make the speed constant for a second, then a different constant for the next second. The distance traveled in each second is the speed in miles per second, and is shown in the graph as the area of a skinny vertical rectangle, of width one second. The areas of the little rectangles under the velocity curve add up to something close to the total distance, and also to the total area under the curve. So we see that in the case of varying speed, as in the case of constant speed, *the distance traveled is equal to the area under the velocity curve.*

To summarize: To a *distance* function $d(t)$ is associated a velocity function $v(t)$, the derivative of $d(t)$. To $v(t)$ in turn



Distance = area under *any* velocity graph (variable speed).

is associated an *area* function $A(t)$, the area under the graph of $v(t)$ up to the vertical line t. *The area $A(t)$ is equal to the distance $d(t)$, the antiderivative of $v(t)$.* The area $A(t)$ under the graph of $v(t)$ is called the "integral" of $v(t)$. The function $d(t)$, from which $v(t)$ was obtained by differentiation, is the antiderivative of $v(t)$. Finding $A(t)$ is called "integrating" $v(t)$. We have just proved the "Fundamental Theorem of Calculus": The area function of v (the integral of v) is equal to the antiderivative of v:

$$A(t) = d(t).$$

We have been thinking of $v(t)$ as a "velocity function." But any function can be interpreted as a velocity function! So the Fundamental Theorem says: The integral of the derivative of any function is the function itself (except possibly for an additive constant).

Computing the derivative directly from its definition is often easy; computing the integral directly from its definition can be hard. The Fundamental Theorem let's us do the hard part by doing the easy part: make a dictionary of differentiation formulas. If in your collection you find a function $w(t)$ whose derivative is $v(t)$, then $w(t)$ is the integral of v!

Let's do an example simple enough that we can get the area by a direct integration. We'll take $v(t) = t$. Not a very realistic velocity function—you'd soon get a speeding ticket! But never mind, this is just theory.

Let's call the region whose area we want to evaluate D. The lower boundary of D is a portion of the positive x-axis; the upper boundary is part of the graph of $v(t) = t$. This graph