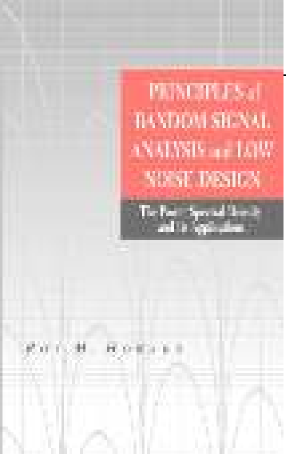
PRINCIPLES of
RANDOM SIGNAL
ANALYSIS and LOW
NOISE DESIGN

Principles of
Random Signal Analysis
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The Power Spectral Density and its Applications

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Contents

Preface

This book gives a systematic account of the Power Spectral Density and details the application of this theory to Communications and Electronics. The level of the book is suited to final year Electrical and Electronic Engineering students, post-graduate students and researchers.

This book arises from the author's research experience in low noise amplifier design and analysis of random processes.

The basis of the book is the definition of the power spectral density using results directly from Fourier theory rather than the more popular approach of defining the power spectral density in terms of the Fourier transform of the autocorrelation function. The difference between use of the two definitions, which are equivalent with an appropriate definition for the autocorrelation function, is that the former greatly facilitates analysis, that is, the determination of the power spectral density of standard signals, as the book demonstrates. The strength, and uniqueness, of the book is that, based on a thorough account of signal theory, it presents a comprehensive and straightforward account of the power spectral density and its application to the important areas of communications and electronics.

The following people have contributed to the book in various ways. First, Prof. J. L. Hullett introduced me to the field of low noise electronic design and has facilitated my career at several important times. Second, Prof. L. Faraone facilitated and supported my research during much of the 1990s. Third, Prof. A. Cantoni, Dr. Y. H. Leung and Prof. K. Fynn supported my research from 1995 to 1997. Fourth, Mr. Nathanael Rensen collaborated on a research project with me over the period 1996 to early 1998. Fifth, Prof. A. Zoubir has provided collegial support and suggested that I contact Dr. P. Meyler from John Wiley & Sons with respect to publication. Sixth, Dr. P. Meyler, Ms. Melissa Yanuzzi and staff at John Wiley have supported and efficiently managed the publication of the book. Finally, several students—undergraduate and postgraduate—have worked on projects related to material in the book and, accordingly, have contributed to the final outcome.

I also wish to thank the staff at Kiribilli Cafe, the Art Gallery Cafe and the King Street Cafe for their indirect support whilst a significant level of editing was in progress. Finally, family and friends will hear a little less about 'The Book' in the future.

Roy M. Howard

December 2001

About the Author

Roy M. Howard has been awarded a BE and a Ph.D. in Electrical Engineering, as well as a BA (Mathematics and Philosophy), by the University of Western Australia. Currently he is a Senior Lecturer in the School of Electrical and Computer Engineering at Curtin University of Technology, Perth, Australia. His research interests include signal theory, stochastic modeling, 1/f noise, low noise amplifier design, nonlinear systems, and nonlinear electronics.

1

Introduction

Random phenomena have their basis in the nature of the physical order (e.g., the nature of electron movement) and limit the performance of many systems including electronic and communication systems. For example, the minimum sensitivity of an amplifier and the distance a signal can be transmitted and recovered, are both limited by random signal variations. On the other hand, there are applications where introduced randomness will enhance aspects of system performance. One example is where a low level randomly varying waveform is added to a repetitive signal to improve the resolution in signal values obtained by an analogue to digital converter and, after averaging (Potzick, 1999; Gray, 1993). Further, in recent years there has been increasing interest in stochastic resonance which occurs when the system response to a weak periodic signal is enhanced by an increase in the level of random variations associated with the system (Luchinsky, 1999; Lindner, 2004).

The importance of random phenomena has led to an increasing number of theoretical results as can be found in books such as Gardner (1990), Papoulis (2002), and Taylor (1998). In communications and electronics a standard way of characterizing random phenomena is through a power spectral density which, for example, facilitates evaluation of the signal to noise ratio of a system operating under prescribed conditions. There are two standard approaches for defining the power spectral density. First, there is a direct Fourier approach. Second, and more commonly, an approach based on the Fourier transform of an autocorrelation function.

With the direct Fourier approach, the power spectral density of a single signal x, for the interval $[0, T]$, is defined as

$$G(T, f) = \frac{|X(T, f)|^2}{T} \tag{1.1}$$

where X is the Fourier transform of x evaluated over the interval $[0, T]$. The alternative approach is to determine the autocorrelation of the signal defined as

$$R(T, t, \tau) = \begin{cases} x(t)x^*(t - \tau) & t \in [0, T], t - \tau \in [0, T] \\ 0 & \text{elsewhere} \end{cases} \tag{1.2}$$

and then take a time average to form an averaged autocorrelation function

$$\bar{R}(T, \tau) = \begin{cases} \dfrac{1}{T} \displaystyle\int_0^{T+\tau} R(T, t, \tau)\, dt & \tau < 0 \\ \dfrac{1}{T} \displaystyle\int_\tau^{T} R(T, t, \tau)\, dt & \tau > 0 \end{cases} \tag{1.3}$$

Finally, the Fourier transform of this function is taken to obtain the power spectral density, that is

$$G(T, f) = \int_{-T}^{T} \bar{R}(T, \tau) e^{-j2\pi f \tau}\, d\tau \tag{1.4}$$

These two approaches lead to identical power spectral density functions when the definitions can be readily generalized for random processes and the infinite time interval. Analytically, the Fourier approach is more direct and leads directly to the interpretation of the power spectral density, at a given frequency, as being proportional to the power in the constituent sinusoidal signal with that frequency. Further, the direct nature of the Fourier approach facilitates the derivation of the power spectral density of signals and random processes.

The following chapters give a systematic account of the theory related to the direct Fourier approach to defining and evaluating the power spectral density. This theory is applied to the derivation of the power spectral density of the random processes commonly encountered in communications and electronics, noise analysis in linear electronic systems, and memoryless transformations of random processes.

Chapter 2 gives appropriate background theory for this book, while Chapter 3 gives a detailed discussion of the two alternative ways the power spectral density can be defined and the equivalence between them. Chapter 4 gives important results that facilitate the evaluation of the power spectral density. Chapters 5 and 6 detail the derivation of the power spectral density of standard random processes encountered in communications and electronics. Chapter 7 details an approach for ascertaining the power spectral density of random processes after a nonlinear memoryless transformation. Chapter 8 discusses the relationship between the input and output signals, and input and output power spectral densities of a linear time invariant system. This chapter gives the necessary background material for Chapter 9, which details the characterization of standard noise signals that occur in electronic devices, and how analysis of such noise signals can be carried out to quantify, and hence minimize, the noise of a linear electronic system.

2

Background: Signal and System Theory

2.1 INTRODUCTION

The power spectral density arises from signal analysis of deterministic signals and random processes, and is required to be evaluated over both the finite and infinite time intervals. While signal analysis for the finite case, for example, the integral on a finite interval of a finite summation of bounded signals, causes few problems, signal analysis for the infinite case is more problematic. For example, it can be the case that the order of the integration and limit operators cannot be interchanged. With the infinite case, careful attention to detail and a reasonable knowledge of underlying mathematical theory is required. Clarity is best achieved for integration, for example, through measure theory and Lebesgue integration.

This chapter gives the necessary mathematical background for the development and application of theory related to the power spectral density that follows in subsequent chapters. First, a review of fundamental results from set theory, real and complex analysis, signal theory and system theory is given. This is followed by an overview of measure and Lebesgue integration, and associated results. Finally, consistent with the requirements of subsequent chapters, results from Fourier theory and a brief introduction to random process theory are given.

2.2 BACKGROUND THEORY

2.2.1 Set Theory

Set theory is fundamental to mathematical analysis, and the following results from set theory are consistent with subsequent analysis. Useful references for set theory include Sprecher (1970), Lipschutz (1998), and Epp (1995).

DEFINITION: SET A set is a collection of distinct entities.

The notation $\{\alpha_1, \alpha_2, \ldots, \alpha_N\}$ is used for the set of distinct entities $\alpha_1, \alpha_2, \ldots, \alpha_N$. The notation $\{x: f(x)\}$ is used for the set of elements x for which the property $f(x)$ is true. The notation $x \in S$ means that the entity denoted x is an element of the set S. The empty set $\{\ \}$ is denoted by $\varnothing$. The complement of a set S, denoted S^C, is defined as $S^C = \{x: x \notin S\}$, where S is usually a subset of a larger set, often the "universal set." The union and intersection of two sets are defined as follows:

$$A \cup B = \{x: x \in A \text{ or } x \in B\}$$
$$A \cap B = \{x: x \in A \text{ and } x \in B\} \tag{2.1}$$

DEFINITION: CHARACTERISTIC FUNCTION OF A SET The characteristic function of a set S is defined according to

$$\chi_S(x) = \begin{cases} 1 & x \in S \\ 0 & x \notin S \end{cases} \tag{2.2}$$

DEFINITION: ORDERED PAIR AND CARTESIAN PRODUCT An ordered pair, denoted (x_1, x_2), where $x_1 \in A$ and $x_2 \in B$, is the set $\{x_1, \{x_1, x_2\}\}$. This definition clearly indicates, for example, that $(x_1, x_2) \neq (x_2, x_1)$ when $x_1 \neq x_2$. The Cartesian product of two sets A and B, denoted $A \times B$, is defined as the set of all possible ordered pairs from these sets, that is

$$A \times B = \{(x, y): x \in A, y \in B\} \tag{2.3}$$

DEFINITION: SUPREMUM AND INFIMUM The supremum of a set A of real numbers, denoted $\sup\{A\}$, is the least upper bound of that set. The infimum of a set A of real numbers, denoted $\inf(A)$, is the greatest lower bound of that set. Formally, $\sup(A)$ is such that (Marsden, 1974, p. 45)

$$\sup(A) \geqslant x \quad \forall x \in A$$
$$\forall \varepsilon > 0 \quad \exists x \in A \quad \text{s.t.} \quad \sup(A) - x < \varepsilon \tag{2.4}$$

Similarly, $\inf(A)$ is such that

$$\inf(A) \leqslant x \quad \forall x \in A$$
$$\forall \varepsilon > 0 \quad \exists x \in A \quad \text{s.t.} \quad x - \inf(A) < \varepsilon \tag{2.5}$$

DEFINITION: PARTITION The set $\{I_1, \ldots, I_N\}$, where $I_i \cap I_j = \varnothing$ for $i \neq j$ and $\bigcup_{i=1}^{N} I_i = I$, is a partition of the set I.

An equivalence relationship generates a partition of a set (Sprecher, 1970, p. 14; Lipson, 1965, p. 552).

Finally, set theory is not without its problems. For example, associated with set theory is Russell's paradox and Cantor's paradox (Lipschutz, 1998 p. [illegible]; Lipschutz, 1998 p. [illegible]).

2.2.2 Real and Complex Analysis

The following gives a review of real and complex analysis consistent with the development of subsequent theory. Useful references for real analysis include Sprecher (1970) and Marsden (1993), while useful references for complex analysis include Marsden (1987) and Brown (1995).

Real analysis has its basis in the natural numbers, denoted $\boldsymbol{N}$ and defined as

$$\boldsymbol{N} = \{1, 2, 3, \ldots\} \tag{2.6}$$

To this set can be added the number zero and the negative of all the numbers in $\boldsymbol{N}$ to form the set of integers, denoted $\boldsymbol{Z}$, that is,

$$\boldsymbol{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} \tag{2.7}$$

The set of positive integers $\boldsymbol{Z}^+$ is defined as being equal to $\boldsymbol{N}$. The set of rational numbers, denoted $\boldsymbol{Q}$, readily follows:

$$\boldsymbol{Q} = \{p/q\colon p, q \in \boldsymbol{Z}, q \neq 0, \gcd(p, q) = 1\} \tag{2.8}$$

where gcd is the greatest common divisor function. The set of rational numbers, however, is not "complete" in the sense that it does not include useful numbers such as the length of the hypotenuse of a right triangle whose sides have unity length, or the area of a circle of unit radius, etc. "Completing" the set of rational numbers to yield the familiar set of real numbers, denoted $\boldsymbol{R}$, can be achieved in two ways. First, through the limit of sequences of rational numbers. Consistent with this approach, a real number can be considered to be the limit of a sequence of rational numbers that converge. For example, the real number 2 is the limit of the sequence $\{2, 2, 2, \ldots\}$, while $\sqrt{2}$ is the limit of the sequence $\{1, 7/5, 141/100, 707/500, \ldots\}$ and so on. Strictly speaking, a real number is an equivalence class associated with a Cauchy sequence of rational numbers (Sprecher, 1970 Ch. 2). Second, through use of a partition (Dedekind cut) of the set of rational numbers into two sets (Dedekind section). The point of partition is associated with a real number (Hall, 19[illegible] p. [illegible]). For example the partition of $\boldsymbol{Q}$ according to

$$\{\{x\colon x \in \boldsymbol{Q}, x \leqslant 0 \text{ or } x^2 < 2\}, \{x\colon x \in \boldsymbol{Q}, x > 0 \text{ and } x^2 > 2\}\} \tag{2.9}$$

defines the real number $\sqrt{2}$.

Algebra on the real numbers is defined through axioms that are of two types (Sprecher, 1970 p. [illegible]; Marsden, 1993 p. 2[illegible]). First, there are "field" axioms that specify the arithmetic operations of addition and multiplication and appropriate additive and multiplicative identity elements. Second, there are "order" axioms that specify the order qualities of real numbers, such as equality, greater than, and less than. The set of real numbers is an "ordered" field.

The set of complex numbers, denoted $\boldsymbol{C}$, is the set of possible ordered pairs that can be generated from real numbers, that is,

$$\boldsymbol{C} = \{(\alpha, \beta): \alpha, \beta \in \boldsymbol{R}\} \tag{2.10}$$

When representing a complex number in the plane the notation $(x, y) = x + jy$ is used where $j = (0, 1)$. The algebra of complex numbers is governed by the rules of vector addition and scalar multiplication, that is

$$\begin{aligned} (x_1, y_1) + (x_2, y_2) &= (x_1 + x_2, y_1 + y_2) \\ a(x_1, y_1) &= (ax_1, ay_1) \quad a \in \boldsymbol{R} \\ (x_1, y_1)(x_2, y_2) &= (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2) \end{aligned} \tag{2.11}$$

From these definitions, the familiar result of $j^2 = -1$, or $j = \sqrt{-1}$, follows.

The conjugate of a complex number (x, y), by definition, is $(x, -y)$.

Definition: Countable and Uncountable Sets   A set is a countable set if each element of the set can be associated, uniquely, with an element of $\boldsymbol{N}$ (Sprecher, 1970, p. [illegible]). If such an association is not possible, then the set is an uncountable set.

The sets $\boldsymbol{N}$, $\boldsymbol{Z}$, and $\boldsymbol{Q}$ are countable sets. The sets $\boldsymbol{R}$ and $\boldsymbol{C}$ are uncountable sets.

Definition. Intervals   If α and β are distinct real numbers with $\alpha < \beta$, then the following sets of points of $\boldsymbol{R}$, denoted intervals, can readily be defined:

$$\begin{aligned} [\alpha, \beta] &= \{x: \alpha \leqslant x \leqslant \beta\} && \text{closed interval} \\ (\alpha, \beta) &= \{x: \alpha < x < \beta\} && \text{open interval} \\ [\alpha, \beta) &= \{x: \alpha \leqslant x < \beta\} && \text{closed/open interval} \\ (\alpha, \beta] &= \{x: \alpha < x \leqslant \beta\} && \text{open/closed interval} \end{aligned} \tag{2.12}$$

Definition: Neighborhood   A neighborhood (NBHD) of a point $x \in \boldsymbol{R}$ is the open interval $(x - \delta, x + \delta)$ where $\delta > 0$ (Sprecher, 1970, p. [illegible]).

Definition: A Contiguous Partition   The set of intervals $\{I_1, \ldots, I_N\}$ is a contiguous partition of the interval I if $\{I_1, \ldots, I_N\}$ is a partition of I and the intervals are ordered such that

$$t \in I_i \Rightarrow t < t_x \qquad \forall t_x \in I_{i+1},\ i \in \{1, \ldots, N-1\} \tag{2.13}$$

2.3 FUNCTIONS, SIGNALS, AND SYSTEMS

Signal and system theory form the basis for a significant level of subsequent analysis. Appropriate definitions and discussion follows. A useful reference for signal theory is Franks (1969).

Definition: Function or Mapping A function, f, is a mapping from a set D, the domain, to a set R, the range, such that only one element in the range is associated with each element in the domain. Such a function is written as $f: D \rightarrow R$. If $y \in R$ and $x \in D$ with x mapping to y under f, then the notation $y = f(x)$ is used (Sprecher, 1970 p. 15).

Note, a function is a special type of relationship between elements from two sets. A "relation," for example, is a more general relationship (Smith, 1990 ch. 5; Pelham, 1990 ch. 1).

Definition: Signal A real and continuous signal is a function from $\boldsymbol{R}$, or a subset of $\boldsymbol{R}$, to $\boldsymbol{R}$, or a subset of $\boldsymbol{R}$. A real and discrete signal is a function from $\boldsymbol{Z}$, or a subset of $\boldsymbol{Z}$, to $\boldsymbol{R}$, or a subset of $\boldsymbol{R}$.

The term "continuous" used here is not related to the concept of continuity. A continuous signal can be represented, for example diagrammatically, as shown in Figure 2.1. Commonly, a real function is implicitly defined by its graph which is a display, for the continuous case, of the set of points $\{(t, f(t)): t \in \boldsymbol{R}\}$. In many instances the variable t denotes time.

A complex signal is a mapping from $\boldsymbol{R}$, or a subset of $\boldsymbol{R}$, to $\boldsymbol{C}$, or a subset of $\boldsymbol{C}$.

Definition: System In the context of engineering, a system is an entity which produces an output signal, usually in response to an input signal, which is transformed in some manner. An autonomous system is one which produces an output signal when there is no input signal. Clock systems and oscillators are examples of autonomous systems.

Definition: Operator A system which produces an output signal in response to an input signal can be modeled by an operator F, as illustrated in

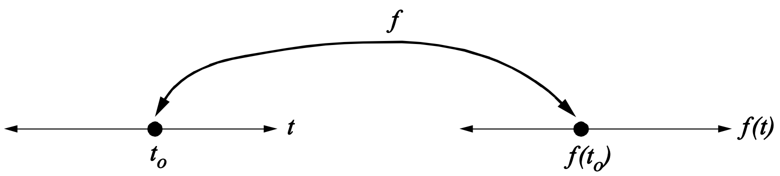

Figure 2.1 *Mapping involved in a continuous real function.*

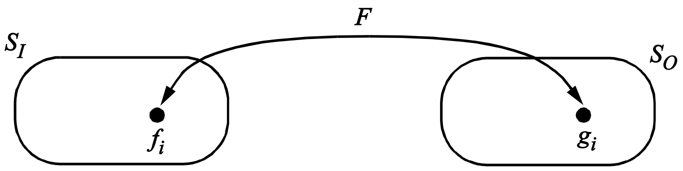

Figure 2.2 *Mapping produced by a system.*

Figure 2.2. In this figure S_I is the set of possible input signals, and S_O is the set of possible output signals. Hence, the operator is a mapping from S_I to S_O, that is, $F: S_I \rightarrow S_O$.

DEFINITION: CONJUGATION OPERATOR A conjugation operator, F_C, is a mapping from the set of complex signals $\{f: \boldsymbol{R} \rightarrow \boldsymbol{C}\}$ to the same set of complex signals, and is defined according to $F_C[f] = f^*$, where $f^*(t) = x(t) - jy(t)$ when $f(t) = x(t) + jy(t)$. Here, the signals x and y are real signals, that is mappings from $\boldsymbol{R}$ to $\boldsymbol{R}$.

2.3.1 Disjoint and Orthogonal Signals

DEFINITION: DISJOINT SIGNALS Two signals $f_1: \boldsymbol{R} \rightarrow \boldsymbol{C}$ and $f_2: \boldsymbol{R} \rightarrow \boldsymbol{C}$ are disjoint on the interval I, if

$$\forall t \in I \qquad f_1(t)\, f_2(t) = 0 \tag{2.14}$$

DEFINITION: SET OF DISJOINT SIGNALS A set of real or complex signals $\{f_1, \ldots, f_N\}$ is a set of disjoint signals on the interval I, if they are pairwise disjoint, that is

$$\forall t \in I,\ i \neq j \qquad f_i(t)\, f_j(t) = 0 \tag{2.15}$$

DEFINITION: ORTHOGONALITY Two signals $f_1: \boldsymbol{R} \rightarrow \boldsymbol{C}$ and $f_2: \boldsymbol{R} \rightarrow \boldsymbol{C}$ are orthogonal on an interval I, if

$$\int_I f_1(t)\, f_2^*(t)\, dt = 0 \tag{2.16}$$

Clearly, disjointness implies orthogonality. Note, orthogonality is defined, in general, via an inner product on elements of an "inner product space" or a Hilbert space (Debnath, 1999 ch. 3; Kreyszig, 1978 ch. 3).

DEFINITION: ORTHOGONAL SET A set of signals $\{f_i: \boldsymbol{R} \to \boldsymbol{C},\ i \in \boldsymbol{Z}^+\}$ is an orthogonal set on an interval I, if the signals are pairwise orthogonal, that is

$$\int_I f_i(t) f_j^*(t)\, dt = 0 \qquad i \neq j \tag{2.17}$$

The most widely used orthogonal sets for an interval $[\alpha, \beta]$ are the sets

$$\left\{1,\ \cos(2\pi i f_o t),\ \sin(2\pi i f_o t):\ i \in \boldsymbol{Z}^+,\ f_o = \frac{1}{\beta - \alpha}\right\} \tag{2.18}$$

$$\left\{e^{j2\pi i f_o t}:\ i \in \boldsymbol{Z},\ f_o = \frac{1}{\beta - \alpha}\right\} \tag{2.19}$$

THEOREM 2.1. SIGNAL DECOMPOSITION *Any signal $f: I \to \boldsymbol{C}$ can be written as the sum of disjoint waveforms, from a disjoint set $\{f_1, \ldots, f_N\}$, according to*

$$f(t) = \sum_{i=1}^{N} f_i(t) \qquad \text{where} \qquad f_i(t) = \begin{cases} f(t) & t \in I_i \\ 0 & \text{elsewhere} \end{cases} \tag{2.20}$$

and $\{I_1, \ldots, I_N\}$ is a partition of I.

Proof. The proof of this result follows directly from the definition of a partition, the definition of set of disjoint waveforms, and by construction.

Signal decomposition using orthogonal basis sets is widely used. A common example is signal decomposition to generate the Fourier series of a signal. Such decomposition is best formulated through use of an inner product on a Hilbert space ([illegible], 1978 ch. 3; [illegible], 1999 ch. [illegible]).

2.3.2 Types of Systems and Operators

The following paragraphs define several types of systems commonly encountered in engineering. In terms of notation, the ith input signal is denoted f_i and the corresponding output signal is denoted g_i.

(a) In general, there may not be an explicit rule defining the mapping between input and output signals produced by a system. In such a case, the relationship between input and output signals can be explicitly stated in a one-to-one manner according to

$$f_1 \to g_1 \qquad f_2 \to g_2 \ \ldots \tag{2.21}$$

(b) Linear systems A linear system is one that can be characterized by an operator L which exhibits the properties of superposition and homogeneity, that is

$$L[\alpha f_i(t) + \beta f_j(t)] = \alpha L[f_i(t)] + \beta L[f_j(t)] \tag{2.22}$$

(c) *Memoryless systems.* A memoryless system is one where the relationship between the input and output signals can be explicitly defined by an operator F, such that

$$g_i = F[f_i] \tag{2.23}$$

An example of such a system is one defined by $F(f) = f^2$ that implies $g_i(t) = f_i^2(t)$.

(d) *Argument altering systems.* Another class of systems is where the relation between input and output signals can be explicitly written in the form

$$g_i(t) = f_i(G[t]) \tag{2.24}$$

for some function G. An example of such a system is a delay system defined by the operator F according to $F[f(t)] = f[G(t)] = f(t - t_d)$, where $G(t) = t - t_d$. Consistent with such a definition $g_i(t) = f_i(t - t_d)$.

(e) Combining the memoryless and argument operators, another class of system can be defined, using an operator F and a function G, according to

$$g_i(t) = F[f_i(G[t])] \tag{2.25}$$

An example of such a system is one where $g_i(t) = f_i^2(t - t_d)$.

(f) A generalization of the memoryless but argument altering system, is one where

$$g_i(t) = \sum_{j=1}^{N} F_j[f_i(G_j[t])] \tag{2.26}$$

An example of such a system is one described by the convolution operator according to

$$g_i(t) = \int_0^t f_i(\lambda)h(t - \lambda)\, d\lambda = \int_0^t f_i(t - \lambda)h(\lambda)\, d\lambda \tag{2.27}$$

As the integral is the limit of a sum, it follows that

$$g_i(t) = \lim_{\Delta t \to 0} \Delta t \sum_{j=1}^{\lfloor t/\Delta t \rfloor} f_i(t - j\Delta t)h(j\Delta t) \tag{2.28}$$

Hence, the convolution can be written as

$$g_i(t) = \lim_{\Delta t \to 0} \Delta t \sum_{j=1}^{\lfloor t/\Delta t \rfloor} F_j[f_i(G_j[t])] \tag{2.29}$$

where $G_j[t] = t - j\Delta t$ and $F_j[f_i] = h(j\Delta t) f_i$.

(g) Implicitly characterized systems. Systems characterized by, for example, differential equations result in implicit operator definitions. For example, consider the system defined by the differential equation

$$\frac{dg_i(t)}{dt} + G[g_i(t)] = F[f_i(t)] \tag{2.30}$$

With D denoting the differentiation operator, the system can be defined as

$$(D + G)(g_i) = F(f_i) \tag{2.31}$$

2.3.3 Defining Output Signal from a Memoryless System

Consider, as shown in Figure 2.3, a memoryless system defined by the operator F. Such an operator can be written in terms of a set of disjoint operators according to

$$F(f) = \sum_{i=1}^{N} F_i(f) \qquad \text{where} \qquad F_i(f) = \begin{cases} F(f) & f \in [f_{i-1}, f_i) \\ 0 & \text{elsewhere} \end{cases} \tag{2.32}$$

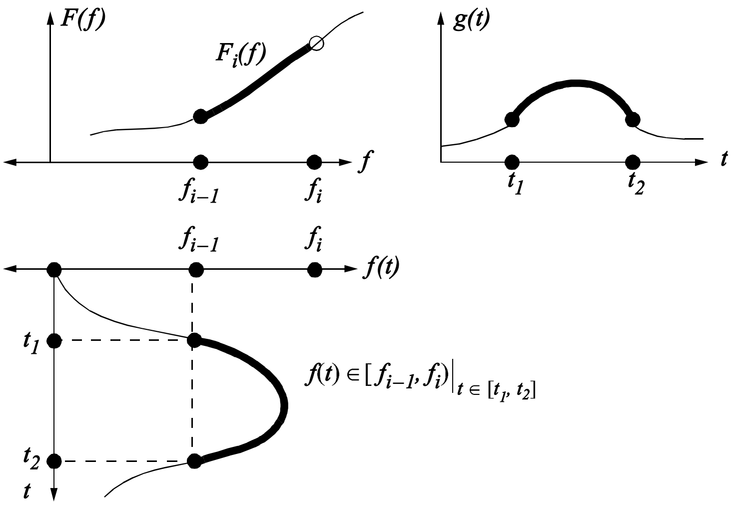

Figure 2.3 Input and output signal of a memoryless system.

The output signal, g, of such a system, in response to an input signal f, can then be determined, consistent with the illustration in Figure 2.5, according to

$$g(t) = F(f(t)) = \sum_{i=1}^{N} F_i(f(t)) \tag{2.33}$$

or in terms of specific time intervals

$$g(t) = \begin{cases} F_1(f(t)) & t \in I_1 \quad I_1 = \{t: f(t) \in [f_0, f_1)\} \\ F_2(f(t)) & t \in I_2 \quad I_2 = \{t: f(t) \in [f_1, f_2)\} \\ \quad \vdots & \end{cases} \tag{2.34}$$

Such a characterization is well-suited to a piecewise linear memoryless system.

2.3.3.1 Decomposition of Output Using Time Partition The input signal, f, to a memoryless nonlinear system can be written, over an interval I, as a summation of disjoint waveforms, that is

$$f(t) = \sum_{i=1}^{N} f_i(t) \qquad f_i(t) = \begin{cases} f(t) & t \in I_i \\ 0 & \text{elsewhere} \end{cases} \tag{2.35}$$

where $\{I_1, \ldots, I_N\}$ is a partition of I. It then follows, by using this partition of I, that the output signal can be written as a summation of disjoint waveforms according to

$$g(t) = \sum_{i=1}^{N} g_i(t) \qquad g_i(t) = \begin{cases} g(t) & t \in I_i \\ 0 & t \notin I_i \end{cases} \tag{2.36}$$

The relationship between the ith disjoint output waveform and the input waveform is

$$g_i(t) = \begin{cases} F(f_i(t)) & t \in I_i \\ 0 & t \notin I_i \end{cases} \tag{2.37}$$

This result is easily proved by noting the following:

$$g_i(t) = \left.\begin{cases} g(t) & t \in I_i \\ 0 & t \notin I_i \end{cases}\right\} = \left.\begin{cases} F(f(t)) & t \in I_i \\ 0 & t \notin I_i \end{cases}\right\} = \begin{cases} F(f_i(t)) & t \in I_i \\ 0 & t \notin I_i \end{cases} \tag{2.38}$$

2.4 SIGNAL PROPERTIES

To establish precise criteria for the validity of various signal relationships related to the power spectral density, precise definitions for basic signal properties such as continuity, differentiability, piecewise smoothness, boundedness, bounded variation, and absolute continuity are required. These properties are detailed in this section. First, however, definitions for signal energy and signal power are given.

DEFINITION: SIGNAL ENERGY AND SIGNAL POWER The energy and average power of a signal $f: \boldsymbol{R} \to \boldsymbol{C}$ on an interval $[\alpha, \beta]$, respectively, are defined as

$$E = \int_{\alpha}^{\beta} |f(t)|^2 \, dt \qquad \bar{P} = \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} |f(t)|^2 \, dt \tag{2.39}$$

2.4.1 Piecewise Continuity and Continuity

DEFINITION: LEFT AND RIGHT HAND CONTINUITY AT A POINT A function is right continuous at a point t_o if the right limit $f(t_o^+)$, defined as follows, exists:

$$f(t_o^+) = \lim_{\delta \to 0} f(t_o + \delta) \qquad \delta > 0 \tag{2.40}$$

Similarly, a function is left continuous at a point t_o if the left limit $f(t_o^-)$, defined as follows, exists:

$$f(t_o^-) = \lim_{\delta \to 0} f(t_o - \delta) \qquad \delta > 0 \tag{2.41}$$

DEFINITION: PIECEWISE CONTINUITY AT A POINT A function f is piecewise continuous at a point t_o if the left and right limits $f(t_o^-)$ and $f(t_o^+)$, exist, that is,

$$\forall \varepsilon > 0 \quad \exists \delta_o > 0 \quad \text{s.t.} \quad 0 < \delta < \delta_o \Rightarrow |f(t_o + \delta) - f(t_o^+)| < \varepsilon \tag{2.42}$$

$$\forall \varepsilon > 0 \quad \exists \delta_o > 0 \quad \text{s.t.} \quad 0 < \delta < \delta_o \Rightarrow |f(t_o^-) - f(t_o - \delta)| < \varepsilon \tag{2.43}$$

and $f(t_o) \in \{f(t_o^-), f(t_o^+)\}$. Here, s.t. is an abbreviation for "such that." The last requirement excludes functions such as

$$f(t) = \begin{cases} \infty & t = t_o \\ k & t \neq t_o \end{cases} \qquad \text{or} \qquad f(t) = \begin{cases} k_o & t = t_o \\ k & t \neq t_o, k \neq k_o \end{cases} \tag{2.44}$$

from being piecewise continuous at t_o.

DEFINITION: PIECEWISE CONTINUITY ON AN INTERVAL A function f is piecewise continuous over an interval I, if it is piecewise continuous at all points in the interval I. For a closed interval $[\alpha, \beta]$ right continuity is required at α while left continuity is required at β.

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