

PRINCIPLES of  
RANDOM SIGNAL  
ANALYSIS and LOW  
NOISE DESIGN

The Power Spectral Density  
and its application

P. P. M. ROELOFFZ

---

*Principles of  
Random Signal Analysis  
and Low Noise Design*

---

# *Principles of Random Signal Analysis and Low Noise Design*

*The Power Spectral Density  
and its Applications*

**Roy M. Howard**

Curtin University of Technology  
Perth, Australia

 **WILEY-  
INTERSCIENCE**

A JOHN WILEY & SONS, INC., PUBLICATION

---

This is printed on acid-free paper. Ⓢ

Copyright © 2002 by John Wiley & Sons, Inc., New York. All rights reserved.

Published simultaneously in Canada.

No part of this publication may be reproduced, stored in a retrieval system or transmitted in any form or by any means, electronic, mechanical, photocopying, recording, scanning or otherwise, except as permitted under Sections 107 or 108 of the 1976 United States Copyright Act, without either the prior written permission of the Publisher, or authorization through payment of the appropriate per-copy fee to the Copyright Clearance Center, 222 Rosewood Drive, Danvers, MA 01923, (978) 750-8400, fax (978) 750-4744. Requests to the Publisher for permission should be addressed to the Permissions Department, John Wiley & Sons, Inc., 605 Third Avenue, New York, NY 10158-0012, (212) 850-6011, fax (212) 850-6008, E-Mail: PERMREQ@WILEY.COM.

For ordering and customer service, call 1-800-CALL-WILEY.

*Library of Congress Cataloging-in-Publication Data is available.*

ISBN: 0-471-22617-3

Printed in the United States of America.

10 9 8 7 6 5 4 3 2 1

---

# Contents

<b>Preface</b>	<b>ix</b>
<b>About the Author</b>	<b>xi</b>
<b>1. Introduction</b>	<b>1</b>
<b>2. Background: Signal and System Theory</b>	<b>3</b>
2.1 Introduction / 3	
2.2 Background Theory / 3	
2.3 Functions, Signals and Systems / 7	
2.4 Signal Properties / 12	
2.5 Measure and Lebesgue Integration / 23	
2.6 Signal Classification / 35	
2.7 Convergence / 36	
2.8 Fourier Theory / 38	
2.9 Random Processes / 44	
2.10 Miscellaneous Results / 45	
Appendix 1: Proof of Theorem 2.11 / 46	
Appendix 2: Proof of Theorem 2.13 / 47	
Appendix 3: Proof of Theorem 2.17 / 47	
Appendix 4: Proof of Theorem 2.27 / 49	
Appendix 5: Proof of Theorem 2.28 / 50	
Appendix 6: Proof of Theorem 2.30 / 52	
Appendix 7: Proof of Theorem 2.31 / 53	
Appendix 8: Proof of Theorem 2.32 / 56	

<b>3. The Power Spectral Density</b>	<b>59</b>
3.1 Introduction / 59	
3.2 Definition / 60	
3.3 Properties / 65	
3.4 Random Processes / 67	
3.5 Existence Criteria / 73	
3.6 Impulsive Case / 74	
3.7 Power Spectral Density via Autocorrelation / 78	
Appendix 1: Proof of Theorem 3.4 / 84	
Appendix 2: Proof of Theorem 3.5 / 85	
Appendix 3: Proof of Theorem 3.8 / 88	
Appendix 4: Proof of Theorem 3.10 / 89	
<b>4. Power Spectral Density Analysis</b>	<b>92</b>
4.1 Introduction / 92	
4.2 Boundedness of Power Spectral Density / 92	
4.3 Power Spectral Density via Signal Decomposition / 95	
4.4 Simplifying Evaluation of Power Spectral Density / 98	
4.5 The Cross Power Spectral Density / 102	
4.6 Power Spectral Density of a Sum of Random Processes / 107	
4.7 Power Spectral Density of a Periodic Signal / 112	
4.8 Power Spectral Density—Periodic Component Case / 119	
4.9 Graphing Impulsive Power Spectral Densities / 122	
Appendix 1: Proof of Theorem 4.2 / 123	
Appendix 2: Proof of Theorem 4.4 / 126	
Appendix 3: Proof of Theorem 4.5 / 128	
Appendix 4: Proof of Theorem 4.6 / 128	
Appendix 5: Proof of Theorem 4.8 / 130	
Appendix 6: Proof of Theorem 4.10 / 132	
Appendix 7: Proof of Theorem 4.11 / 134	
Appendix 8: Proof of Theorem 4.12 / 136	
<b>5. Power Spectral Density of Standard Random Processes—Part 1</b>	<b>138</b>
5.1 Introduction / 138	
5.2 Signaling Random Processes / 138	
5.3 Digital to Analogue Converter Quantization / 152	
5.4 Jitter / 155	
5.5 Shot Noise / 160	
5.6 Generalized Signaling Processes / 166	
Appendix 1: Proof of Theorem 5.1 / 168	
Appendix 2: Proof of Theorem 5.2 / 171	
Appendix 3: Proof of Equation 5.73 / 173	
Appendix 4: Proof of Theorem 5.3 / 174	

Appendix 5: Proof of Theorem 5.4 / 176  
Appendix 6: Proof of Theorem 5.5 / 177

**6. Power Spectral Density of Standard Random Processes—Part 2** **179**

- 6.1 Introduction / 179
- 6.2 Sampled Signals / 179
- 6.3 Quadrature Amplitude Modulation / 185
- 6.4 Random Walks / 192
- 6.5 1/f Noise / 198
  - Appendix 1: Proof of Theorem 6.1 / 200
  - Appendix 2: Proof of Theorem 6.2 / 201
  - Appendix 3: Proof of Theorem 6.3 / 202
  - Appendix 4: Proof of Equation 6.39 / 204

**7. Memoryless Transformations of Random Processes** **206**

- 7.1 Introduction / 206
- 7.2 Power Spectral Density after a Memoryless Transformation / 206
- 7.3 Examples / 211
  - Appendix 1: Proof of Theorem 7.1 / 223
  - Appendix 2: Fourier Results for Raised Cosine Frequency Modulation / 224

**8. Linear System Theory** **229**

- 8.1 Introduction / 229
- 8.2 Impulse Response / 230
- 8.3 Input-Output Relationship / 232
- 8.4 Fourier and Laplace Transform of Output / 232
- 8.5 Input-Output Power Spectral Density Relationship / 238
- 8.6 Multiple Input-Multiple Output Systems / 243
  - Appendix 1: Proof of Theorem 8.1 / 246
  - Appendix 2: Proof of Theorem 8.2 / 248
  - Appendix 3: Proof of Theorem 8.3 / 249
  - Appendix 4: Proof of Theorem 8.4 / 251
  - Appendix 5: Proof of Theorem 8.6 / 252
  - Appendix 6: Proof of Theorem 8.7 / 253
  - Appendix 7: Proof of Theorem 8.8 / 255

**9. Principles of Low Noise Electronic Design** **256**

- 9.1 Introduction / 256
- 9.2 Gaussian White Noise / 259
- 9.3 Standard Noise Sources / 264
- 9.4 Noise Models for Standard Electronic Devices / 266
- 9.5 Noise Analysis for Linear Time Invariant Systems / 269

9.6 Input Equivalent Current and Voltage Sources / 278  
9.7 Transferring Noise Sources / 282  
9.8 Results for Low Noise Design / 285  
9.9 Noise Equivalent Bandwidth / 285  
9.10 Power Spectral Density of a Passive Network / 287  
    Appendix 1: Proof of Theorem 9.2 / 291  
    Appendix 2: Proof of Theorem 9.4 / 294  
    Appendix 3: Proof of Conjecture for Ladder Structure / 296

**Notation** 300  
**References** 302  
**Index** 307



---

# *Preface*

This book gives a systematic account of the Power Spectral Density and details the application of this theory to Communications and Electronics. The level of the book is suited to final year Electrical and Electronic Engineering students, post-graduate students and researchers.

This book arises from the author's research experience in low noise amplifier design and analysis of random processes.

The basis of the book is the definition of the power spectral density using results directly from Fourier theory rather than the more popular approach of defining the power spectral density in terms of the Fourier transform of the autocorrelation function. The difference between use of the two definitions, which are equivalent with an appropriate definition for the autocorrelation function, is that the former greatly facilitates analysis, that is, the determination of the power spectral density of standard signals, as the book demonstrates. The strength, and uniqueness, of the book is that, based on a thorough account of signal theory, it presents a comprehensive and straightforward account of the power spectral density and its application to the important areas of communications and electronics.

The following people have contributed to the book in various ways. First, Prof. J. L. Hullett introduced me to the field of low noise electronic design and has facilitated my career at several important times. Second, Prof. L. Faraone facilitated and supported my research during much of the 1990s. Third, Prof. A. Cantoni, Dr. Y. H. Leung and Prof. K. Fynn supported my research from 1995 to 1997. Fourth, Mr. Nathanael Rensen collaborated on a research project with me over the period 1996 to early 1998. Fifth, Prof. A. Zoubir has provided collegial support and suggested that I contact Dr. P. Meyler from John Wiley & Sons with respect to publication. Sixth, Dr. P. Meyler,

Ms. Melissa Yanuzzi and staff at John Wiley have supported and efficiently managed the publication of the book. Finally, several students—undergraduate and postgraduate—have worked on projects related to material in the book and, accordingly, have contributed to the final outcome.

I also wish to thank the staff at Kiribilli Cafe, the Art Gallery Cafe and the King Street Cafe for their indirect support whilst a significant level of editing was in progress. Finally, family and friends will hear a little less about ‘The Book’ in the future.

ROY M. HOWARD

*December 2001*

---

## *About the Author*

Roy M. Howard has been awarded a BE and a Ph.D. in Electrical Engineering, as well as a BA (Mathematics and Philosophy), by the University of Western Australia. Currently he is a Senior Lecturer in the School of Electrical and Computer Engineering at Curtin University of Technology, Perth, Australia. His research interests include signal theory, stochastic modeling, 1/f noise, low noise amplifier design, nonlinear systems, and nonlinear electronics.

---

# 1

---

## *Introduction*

Random phenomena have their base in the nature of the physical world (e.g., the motion of electrons in a wire) and limit the performance of many systems including electronic and communication systems. For example, the minimum sensitivity of an amplifier and the distance a signal can be transmitted and recovered, are both limited by random signal variations. On the other hand, there are applications where introduced randomness will enhance aspects of system performance. One example is where a low level randomly varying waveform is added to a repetitive signal to improve the resolution in signal values obtained by an analogue to digital converter and after averaging (Potzick, 1999; Gray, 1999). Further, in recent years there has been increasing interest in stochastic resonance which occurs when the system response to a weak periodic signal is enhanced by an increase in the level of random variations associated with the system (Luchinsky, 1999; Leung, 2004).

The importance of random phenomena has led to an increasing number of theoretical results as can be found in books such as Granger (1990), Papoulis (2002), and Taylor (1998). In communications and electronics a standard way of characterizing random phenomena is through a power spectral density which, for example, facilitates comparison of the signal to noise ratio of a system operating under prescribed conditions. There are two standard approaches for defining the power spectral density. First, there is a direct Fourier approach, second, and more commonly, an approach based on the Fourier transform of an autocorrelation function.

With the direct Fourier approach, the power spectral density of a single signal  $x$ , for the interval  $[0, T]$ , is defined as

$$G(T, f) = \frac{|X(T, f)|^2}{T} \quad (1.1)$$

where  $X$  is the Fourier transform of  $x$  evaluated over the interval  $[0, T]$ . The alternative approach is to determine the autocorrelation of the signal

defined as

$$R(T, t, \tau) = \begin{cases} x(t)x^*(t - \tau) & t \in [0, T], t - \tau \in [0, T] \\ 0 & \text{elsewhere} \end{cases} \quad (1.2)$$

and then takes a time average to form an averaged autocorrelation function

$$\bar{R}(T, \tau) = \begin{cases} \frac{1}{T} \int_0^{T+\tau} R(T, t, \tau) dt & \tau < 0 \\ \frac{1}{T} \int_{\tau}^T R(T, t, \tau) dt & \tau > 0 \end{cases} \quad (1.3)$$

Finally, the Fourier transform of this function is taken to obtain the power spectral density, that is

$$G(T, f) = \int_{-T}^T \bar{R}(T, \tau) e^{-j2\pi f\tau} d\tau \quad (1.4)$$

These two approaches lead to identical power spectral density functions where the definitions can be readily generalized for random processes and the infinite time interval. Analytically, the Fourier approach is more direct and leads directly to the interpretation of the power spectral density at a given frequency, as being proportional to the power in the constituent sinusoidal signal with that frequency. Further, the direct nature of the Fourier approach facilitates the derivation of the power spectral density of signals and random processes.

The following chapters give a systematic account of the theory related to the direct Fourier approach to defining and evaluating the power spectral density. This theory is applied to the derivation of the power spectral density of the random processes commonly encountered in communications and electronics, noise analysis in linear electronic systems, and instantaneous characterizations of random processes.

Chapter 2 gives appropriate background theory for this book, while Chapter 3 gives a detailed discussion of the two alternative ways the power spectral density can be defined and the equivalence between them. Chapter 4 gives important results that facilitate the derivation of the power spectral density. Chapter 5 and 6 detail the derivation of the power spectral density of standard random processes encountered in communications and electronics. Chapter 7 details an approach for ascertaining the power spectral density of random processes after a nonlinear memoryless transformation. Chapter 8 discusses the relationship between the input and output signals, and input and output power spectral densities of a linear time-invariant system. This chapter gives the necessary background material for Chapter 9, which details the characterization of standard noise signals that occur in electronic devices, and how analysis of such noise signals can be carried out to quantify and hence minimize the noise of a linear electronic system.

---

# 2

---

## *Background: Signal and System Theory*

### 2.1 INTRODUCTION

The power spectral density arises from signal analysis of deterministic signals and random processes, and is required to be evaluated over both the finite and infinite time intervals. While signal analysis on the finite case, for example, the integral on a finite interval of a finite summation of bounded signals, causes few problems, signal analysis for the infinite case is more problematic. For example, it can be the case that the order of the integration and limit operators cannot be interchanged. With the infinite case, careful attention to detail and a reasonable knowledge of underlying mathematical theory is required. Clarity is best achieved, for integration, for example, through measure theory and Lebesgue integration.

This chapter gives the necessary mathematical background for the development and application of theory related to the power spectral density that follows in subsequent chapters. First, a review of fundamental results from set theory, real and complex analysis, signal theory and system theory is given. This is followed by an overview of measure and Lebesgue integration, and associated results. Finally, consistent with the requirements of subsequent chapters, results from Fourier theory and a brief introduction to random process theory are given.

### 2.2 BACKGROUND THEORY

#### 2.2.1 Set Theory

Set theory is fundamental to mathematical analysis, and the following results from set theory are consistent with subsequent analysis. Useful references for set theory include Spivak (1970), Lipschutz (1998), and Epp (1995).

**DEFINITION: SET** A set is a collection of distinct entities.

The notation  $\{\alpha_1, \alpha_2, \dots, \alpha_N\}$  is used for the set of distinct entities  $\alpha_1, \alpha_2, \dots, \alpha_N$ . The notation  $\{x: f(x)\}$  is used for the set of elements  $x$  for which the property  $f(x)$  is true. The notation  $x \in S$  means that the entity denoted  $x$  is an element of the set  $S$ . The empty set  $\{\}$  is denoted by  $\emptyset$ . The complement of a set  $S$ , denoted  $S^c$ , is defined as  $S^c = \{x: x \notin S\}$ , where  $S$  is usually a subset of a larger set (often the "universal set"). The union and intersection of two sets are defined as follows:

$$\begin{aligned} A \cup B &= \{x: x \in A \text{ or } x \in B\} \\ A \cap B &= \{x: x \in A \text{ and } x \in B\} \end{aligned} \quad (2.1)$$

**DEFINITION: CHARACTERISTIC FUNCTION OF A SET** The characteristic function of a set  $S$  is defined as follows:

$$\chi_S(x) = \begin{cases} 1 & x \in S \\ 0 & x \notin S \end{cases} \quad (2.2)$$

**DEFINITION: ORDERED PAIR AND CARTESIAN PRODUCT** An ordered pair, denoted  $(x_1, x_2)$ , where  $x_1 \in A$  and  $x_2 \in B$ , is the set  $\{x_1, \{x_1, x_2\}\}$ . This definition clearly indicates, for example, that  $(x_1, x_2) \neq (x_2, x_1)$  when  $x_1 \neq x_2$ . The Cartesian product of two sets  $A$  and  $B$ , denoted  $A \times B$ , is defined as the set of all possible ordered pairs from these sets, that is

$$A \times B = \{(x, y): x \in A, y \in B\} \quad (2.3)$$

**DEFINITION: SUPREMUM AND INFIMUM** The supremum of a set  $A$  of real numbers, denoted  $\sup\{A\}$ , is the least upper bound of that set. The infimum of a set  $A$  of real numbers, denoted  $\inf\{A\}$ , is the greatest lower bound of that set (usually,  $\sup\{A\}$  is written  $\sup A$ ) (Morse and Morse, 1938, p. 4).

$$\begin{aligned} \sup(A) &\geq x \quad \forall x \in A \\ \forall \varepsilon > 0 \quad \exists x \in A \quad \text{s.t.} \quad \sup(A) - x < \varepsilon \end{aligned} \quad (2.4)$$

Similarly,  $\inf(A)$  is such that

$$\begin{aligned} \inf(A) &\leq x \quad \forall x \in A \\ \forall \varepsilon > 0 \quad \exists x \in A \quad \text{s.t.} \quad x - \inf(A) < \varepsilon \end{aligned} \quad (2.5)$$

**DEFINITION: PARTITION** The set  $\{I_1, \dots, I_N\}$ , where  $I_i \cap I_j = \emptyset$  for  $i \neq j$  and  $\bigcup_{i=1}^N I_i = I$ , is a partition of the set  $I$ .

An equivalence relationship generates a partition of a set (Spivak, 1976, p. 4; Lipsitz, 1993, p. 538).

Finally, set theory is not without its problems. For example, associated with set theory is Russell's paradox and Cantor's paradox (Leptz, 1995, p. 208; Lipschitz, 1998, p. 112).

## 2.2.2 Real and Complex Analysis

The following gives a review of real and complex analysis consistent with the development of subsequent theory. Useful references for real analysis include Sprecher (1970) and Marsden (1993), while useful references for complex analysis include Marsden (1987) and Brown (1995).

Real analysis has its basis in the natural numbers, denoted  $\mathbf{N}$  and defined as

$$\mathbf{N} = \{1, 2, 3, \dots\} \quad (2.6)$$

To this set can be added the number zero and the negative of all the numbers in  $\mathbf{N}$  to form the set of integers, denoted  $\mathbf{Z}$ , that is,

$$\mathbf{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\} \quad (2.7)$$

The set of positive integers  $\mathbf{Z}^+$  is defined as being equal to  $\mathbf{N}$ . The set of rational numbers, denoted  $\mathbf{Q}$  readily follows

$$\mathbf{Q} = \{p/q; p, q \in \mathbf{Z}, q \neq 0, \text{gcd}(p, q) = 1\} \quad (2.8)$$

where gcd is the greatest common divisor function. The set of rational numbers, however, is not "complete" in the sense that it does not include useful numbers such as the length of the hypotenuse of a right triangle whose sides have only length 1, or the area of a circle of unit radius, and "completing" the set of rational numbers to yield the familiar set of real numbers, denoted  $\mathbf{R}$ , can be achieved in two ways. First, through the limit of sequences of rational numbers. Consistent with this approach, a real number can be considered to be the limit of a sequence of rational numbers that converges. For example, the real number 2 is the limit of the sequence  $\{2, 2, 2, \dots\}$ , while  $\sqrt{2}$  is the limit of the sequence  $\{1, \sqrt{2}, 1.4142, 1.41421, 1.414213, \dots\}$  and so on. Strictly speaking, a real number is an equivalence class associated with a Cauchy sequence of rational numbers (Sprecher, 1970, Ch. 1). Second, through use of a partition (Dedekind cut) of the set of rational numbers into two sets (Dedekind, see (19)). The point of partition is associated with a real number (Kull, 1993, p. 18). For example, the partition of  $\mathbf{Q}$  according to

$$\{\{x; x \in \mathbf{Q}, x \leq 0 \text{ or } x^2 < 2\}, \{x; x \in \mathbf{Q}, x > 0 \text{ and } x^2 > 2\}\} \quad (2.9)$$

defines the real number  $\sqrt{2}$ .

Algebra on the real numbers is defined through axioms that are of two types (Sprecher, 1970, p. 17; Marsden, 1993, p. 28). First, there are "field" axioms that



specify the arithmetic operations of addition and multiplication are appropriate additive and multiplicative identity elements. Beyond these are "order relations" that specify the order qualities of real numbers, such as equality, greater than, and less than. The set of real numbers is an ordered field.

The set of complex numbers, denoted  $\mathbf{C}$ , is the set of possible ordered pairs that can be generated from real numbers. That is,

$$\mathbf{C} = \{(\alpha, \beta): \alpha, \beta \in \mathbf{R}\} \quad (2.10)$$

When representing a complex number in the plane the notation  $(x, y) = x + jy$  is used where  $j = (0, 1)$ . The algebra of complex numbers is governed by the rules of vector addition and scalar multiplication, that is

$$\begin{aligned} (x_1, y_1) + (x_2, y_2) &= (x_1 + x_2, y_1 + y_2) \\ a(x_1, y_1) &= (ax_1, ay_1) \quad a \in \mathbf{R} \\ (x_1, y_1)(x_2, y_2) &= (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2) \end{aligned} \quad (2.11)$$

From these definitions, the familiar result  $j^2 = -1$ , or  $j = \sqrt{-1}$ , follows.

The conjugate of a complex number  $(x, y)$  is defined as  $(x, -y)$ .

**DEFINITION: COUNTABLE AND UNCOUNTABLE SETS** A set is countable if each element of the set can be associated uniquely with an element of  $\mathbf{N}$  (Sprotter, 1996 p. 12). If such an association is not possible, then the set is an uncountable set.

The sets  $\mathbf{N}$ ,  $\mathbf{Z}$ , and  $\mathbf{Q}$  are countable sets. The sets  $\mathbf{R}$  and  $\mathbf{C}$  are uncountable sets.

**DEFINITION: INTERVALS** If  $\alpha$  and  $\beta$  are distinct real numbers with  $\alpha < \beta$ , then the following sets of points of  $\mathbf{R}$ , denoted intervals, can readily be defined:

$$\begin{aligned} [\alpha, \beta] &= \{x: \alpha \leq x \leq \beta\} && \text{closed interval} \\ (\alpha, \beta) &= \{x: \alpha < x < \beta\} && \text{open interval} \\ [\alpha, \beta) &= \{x: \alpha \leq x < \beta\} && \text{closed/open interval} \\ (\alpha, \beta] &= \{x: \alpha < x \leq \beta\} && \text{open/closed interval} \end{aligned} \quad (2.12)$$

**DEFINITION: NEIGHBORHOOD** A neighborhood (NBHD) of a point  $x \in \mathbf{R}$  is the open interval  $(x - \delta, x + \delta)$  where  $\delta > 0$  (Sprotter, 1996 p. 15).

**DEFINITION: A CONTIGUOUS PARTITION** The set of intervals  $\{I_1, \dots, I_N\}$  is a contiguous partition of the interval  $I$  if  $\{I_1, \dots, I_N\}$  is a partition of  $I$  and the intervals are ordered such that

$$t \in I_i \Rightarrow t < t_x \quad \forall t_x \in I_{i+1}, i \in \{1, \dots, N-1\} \quad (2.13)$$

## 2.3 FUNCTIONS, SIGNALS, AND SYSTEMS

Signal and system theory form the basis for a significant level of subsequent analysis. Appropriate definitions and discussion follows. A useful reference for signal theory is Tracks (1969).

**DEFINITION: FUNCTION OR MAPPING** A function,  $f$ , is a mapping from a set  $D$ , the domain, to a set  $R$ , the range, such that only one element in the range is associated with each element in the domain. Such a function is written as  $f: D \rightarrow R$ . If  $y \in R$  and  $x \in D$  with  $x$  mapping to  $y$  under  $f$ , then the notation  $y = f(x)$  is used (Sprockel, 1970 p. 13).

Note, a function is a special type of relationship between elements from two sets. A relation, for example, is a more general relationship (Smith, 1990 ch. 3, Peilichon, 1970 ch. 1).

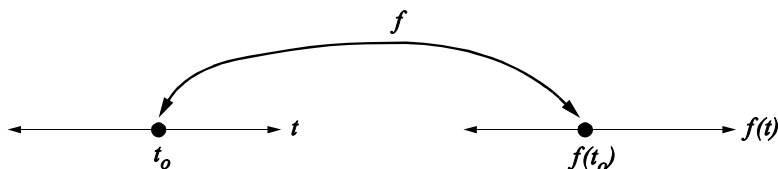
**DEFINITION: SIGNAL** A real and continuous signal is a function from  $\mathbf{R}$ , or a subset of  $\mathbf{R}$ , to  $\mathbf{R}$ , or a subset of  $\mathbf{R}$ . A real and discrete signal is a function from  $\mathbf{Z}$ , or a subset of  $\mathbf{Z}$ , to  $\mathbf{R}$ , or a subset of  $\mathbf{R}$ .

The term "continuous" used here is not related to the concept of continuity. A continuous signal can be represented, for example, diagrammatically, as shown in Figure 2.1. Typically, a real function is implicitly defined by its graph which is a display, for the continuous case, of the set of points  $\{(t, f(t)): t \in \mathbf{R}\}$ . In many instances the variable  $t$  denotes time.

A complex signal is a mapping from  $\mathbf{R}$ , or a subset of  $\mathbf{R}$ , to  $\mathbf{C}$ , or a subset of  $\mathbf{C}$ .

**DEFINITION: SYSTEM** In the context of engineering, a system is an entity which produces an output signal usually in response to an input signal which is transformed in some manner. An autonomous system is one which produces an output signal when there is no input signal. Control systems and regulators are examples of autonomous systems.

**DEFINITION: OPERATOR** A system which produces an output signal in response to an input signal can be modeled by an operator  $F$ , as illustrated in



**Figure 2.1** Mapping involved in a continuous real function.

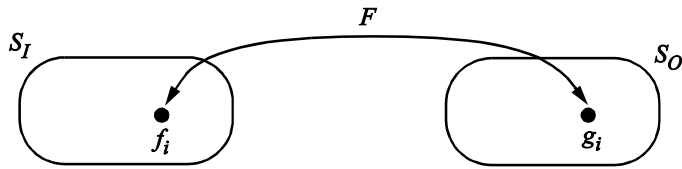


Figure 2.2 Mapping produced by a system.

Figure 2.2. In this figure  $S_I$  is the set of possible input signals, and  $S_O$  is the set of possible output signals. Hence, the operator is a mapping from  $S_I$  to  $S_O$ , that is,  $F: S_I \rightarrow S_O$ .

**DEFINITION: CONJUGATION OPERATOR** A conjugation operator,  $F_C$ , is a mapping from the set of complex signals  $\{f: \mathbf{R} \rightarrow \mathbf{C}\}$  to the same set of complex signals, and is defined according to  $F_C[f] = f^*$ , where  $f^*(t) = x(t) - jy(t)$  when  $f(t) = x(t) + jy(t)$ . Here, the signals  $x$  and  $y$  are real signals, that is, elements from  $\mathbf{R}$  to  $\mathbf{R}$ .

### 2.3.1 Disjoint and Orthogonal Signals

**DEFINITION: DISJOINT SIGNALS** Two signals  $f_1: \mathbf{R} \rightarrow \mathbf{C}$  and  $f_2: \mathbf{R} \rightarrow \mathbf{C}$  are disjoint on the interval  $I$ , if

$$\forall t \in I \quad f_1(t)f_2(t) = 0 \quad (2.14)$$

**DEFINITION: SET OF DISJOINT SIGNALS** A set of real or complex signals  $\{f_1, \dots, f_N\}$  is a set of disjoint signals on the interval  $I$ , if they are pairwise disjoint, that is

$$\forall t \in I, i \neq j \quad f_i(t)f_j(t) = 0 \quad (2.15)$$

**DEFINITION: ORTHOGONALITY** Two signals  $f_1: \mathbf{R} \rightarrow \mathbf{C}$  and  $f_2: \mathbf{R} \rightarrow \mathbf{C}$  are orthogonal on the interval  $I$ , if

$$\int_I f_1(t)f_2^*(t) dt = 0 \quad (2.16)$$

Clearly, disjointness implies orthogonality. Note, orthogonality is defined, in general, for an inner product on elements of the "inner product space" of a Hilbert space (Lichtenberg, 1979 ch. 3; Kreyszig, 1978 ch. 1).

**DEFINITION: ORTHOGONAL SET** A set of signals  $\{f_i: \mathbf{R} \rightarrow \mathbf{C}, i \in \mathbf{Z}^+\}$  is an orthogonal set on an interval  $I$ , if the signals are pairwise orthogonal, that is

$$\int_I f_i(t) f_j^*(t) dt = 0 \quad i \neq j \quad (2.17)$$

The most widely used orthogonal sets on an interval  $[\alpha, \beta]$  are the sets

$$\left\{ 1, \cos(2\pi i f_o t), \sin(2\pi i f_o t): i \in \mathbf{Z}^+, f_o = \frac{1}{\beta - \alpha} \right\} \quad (2.18)$$

$$\left\{ e^{i2\pi i f_o t}: i \in \mathbf{Z}, f_o = \frac{1}{\beta - \alpha} \right\} \quad (2.19)$$

**THEOREM 2.1. SIGNAL DECOMPOSITION** Any signal  $f: I \rightarrow \mathbf{C}$  can be written as the sum of signals belonging to a finite set  $\{f_1, \dots, f_N\}$ , according to

$$f(t) = \sum_{i=1}^N f_i(t) \quad \text{where} \quad f_i(t) = \begin{cases} f(t) & t \in I_i \\ 0 & \text{elsewhere} \end{cases} \quad (2.20)$$

and  $\{I_1, \dots, I_N\}$  is a partition of  $I$ .

*Proof.* The proof of this result follows directly from the definition of a partition, the definition of set of disjoint intervals, and by construction.

Signal decomposition using orthogonal basis sets is widely used. A common example is signal decomposition to generate the Fourier series of a signal. Such decomposition is best formulated through use of an inner product on a Hilbert space (Arovizki, 1978 and A. Papoulis, 1959, ch. 1).

### 2.3.2 Types of Systems and Operators

The following paragraphs define several types of systems commonly encountered in engineering. In terms of notation, the  $i$ th input signal is denoted  $f_i$  and the corresponding output signal is denoted  $g_i$ .

(a) In general, there may not be an explicit rule defining the mapping between input and output signals produced by a system. In such a case, the relationship between input and output signals can be explicitly stated in a one-to-one manner according to

$$f_1 \rightarrow g_1 \quad f_2 \rightarrow g_2 \dots \quad (2.21)$$

(b) **Linear systems** A linear system is one that can be characterized by an operator  $L$  which exhibits the properties of superposition and

homogeneity, that is

$$L[\alpha f_i(t) + \beta f_j(t)] = \alpha L[f_i(t)] + \beta L[f_j(t)] \quad (2.22)$$

(c) **Memoryless systems** A memoryless system is one where the relationship between the input and output signals can be explicitly defined by an operator  $F$ , such that

$$g_i = F[f_i] \quad (2.23)$$

An example of such a system is one defined by  $F(f) = f^2$  that implies  $g_i(t) = f_i^2(t)$ .

(d) **Alignment altering systems** Another class of systems is where the relation between input and output signals can be explicitly written in the form

$$g_i(t) = f_i(G[t]) \quad (2.24)$$

for some function  $G$ . An example of such a system is a delay system defined by the operator  $F$  according to  $F[f(t)] = f[G(t)] = f(t - t_d)$ , where  $G(t) = t - t_d$ . Consistent with such a definition  $g_i(t) = f_i(t - t_d)$ .

(e) Combining the memoryless and alignment operators, another class of system can be defined, using an operator  $F$  and a function  $G$ , according to

$$g_i(t) = F[f_i(G[t])] \quad (2.25)$$

An example of such a system is one where  $g_i(t) = f_i^2(t - t_d)$ .

(f) A generalization of the memoryless but alignment altering system, is one where

$$g_i(t) = \sum_{j=1}^N F_j[f_i(G_j[t])] \quad (2.26)$$

An example of such a system is one described by the convolution operator according to

$$g_i(t) = \int_0^t f_i(\lambda)h(t - \lambda) d\lambda = \int_0^t f_i(t - \lambda)h(\lambda) d\lambda \quad (2.27)$$

As the integral is the limit of a sum, it follows that

$$g_i(t) = \lim_{\Delta t \rightarrow 0} \Delta t \sum_{j=1}^{\lfloor t/\Delta t \rfloor} f_i(t - j\Delta t)h(j\Delta t) \quad (2.28)$$

Hence, the convolution can be written as

$$g_i(t) = \lim_{\Delta t \rightarrow 0} \Delta t \sum_{j=1}^{\lfloor t/\Delta t \rfloor} F_j[f_i(G_j[t])] \tag{2.29}$$

where  $G_j[t] = t - j\Delta t$  and  $F_j[f_i] = h(j\Delta t)f_i$ .

(g) Impulses characterized systems (systems characterized by JCT multiple differential equations) result in impulse operator definitions. For example, consider the system defined by the differential equation

$$\frac{dg_i(t)}{dt} + G[g_i(t)] = F[f_i(t)] \tag{2.30}$$

With  $D$  denoting the differentiation operator, the system can be defined as

$$(D + G)(g_i) = F(f_i) \tag{2.31}$$

### 2.3.3 Defining Output Signal from a Memoryless System

Further, as shown in Figure 2.3, a memoryless system defined by the operator  $F$  through an operator can be written in terms of a set of disjoint operators according to

$$F(f) = \sum_{i=1}^N F_i(f) \quad \text{where} \quad F_i(f) = \begin{cases} F(f) & f \in [f_{i-1}, f_i] \\ 0 & \text{elsewhere} \end{cases} \tag{2.32}$$

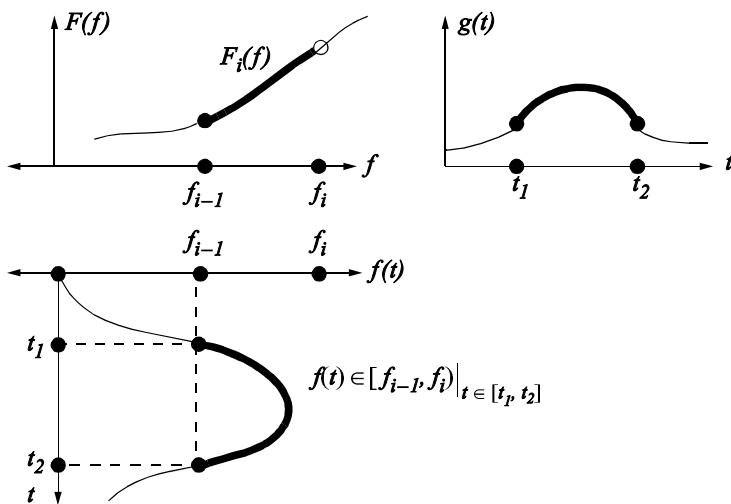


Figure 2.3 Input and output signal of a memoryless system.

The output signal  $g$  of such a system, in response to an input signal  $f$ , can then be determined, consistent with the discussion in Figure 2.3, according to

$$g(t) = F(f(t)) = \sum_{i=1}^N F_i(f(t)) \quad (2.33)$$

or in terms of specific time intervals

$$g(t) = \begin{cases} F_1(f(t)) & t \in I_1 \quad I_1 = \{t: f(t) \in [f_0, f_1]\} \\ F_2(f(t)) & t \in I_2 \quad I_2 = \{t: f(t) \in [f_1, f_2]\} \\ \vdots & \vdots \end{cases} \quad (2.34)$$

Such a characterization is well-suited to a piecewise linear memoryless system.

**2.3.3.1 Decomposition of Output Using Time Partition** The input signal,  $f$ , to a memoryless nonlinear system can be written over an interval  $I$ , as a summation of disjoint waveforms, that is

$$f(t) = \sum_{i=1}^N f_i(t) \quad f_i(t) = \begin{cases} f(t) & t \in I_i \\ 0 & \text{elsewhere} \end{cases} \quad (2.35)$$

where  $\{I_1, \dots, I_N\}$  is a partition of  $I$ . Then, likewise, by using this partition of  $I$ , that the output signal can be written as a summation of disjoint waveforms, according to

$$g(t) = \sum_{i=1}^N g_i(t) \quad g_i(t) = \begin{cases} g(t) & t \in I_i \\ 0 & t \notin I_i \end{cases} \quad (2.36)$$

The relationship between the  $i$ th disjoint output waveform and the total waveform is

$$g_i(t) = \begin{cases} F(f_i(t)) & t \in I_i \\ 0 & t \notin I_i \end{cases} \quad (2.37)$$

This result is easily proved by noting that

$$g_i(t) = \begin{cases} g(t) & t \in I_i \\ 0 & t \notin I_i \end{cases} = \begin{cases} F(f(t)) & t \in I_i \\ 0 & t \notin I_i \end{cases} = \begin{cases} F(f_i(t)) & t \in I_i \\ 0 & t \notin I_i \end{cases} \quad (2.38)$$

## 2.4 SIGNAL PROPERTIES

To establish precise criteria on the validity of various signal relationships related to the power spectral density, precise definitions for basic signal

properties such as continuity, differentiability, piecewise smoothness, boundedness, bounded variation, and absolute continuity are required. These properties are detailed in this section. First, however, definitions for signal energy and signal power are given.

**DEFINITION: SIGNAL ENERGY AND SIGNAL POWER** The energy and average power of a signal  $f: \mathbf{R} \rightarrow \mathbf{C}$  on an interval  $[\alpha, \beta]$ , respectively, are defined by

$$E = \int_{\alpha}^{\beta} |f(t)|^2 dt \quad \bar{P} = \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} |f(t)|^2 dt \quad (2.39)$$

### 2.4.1 Piecewise Continuity and Continuity

**DEFINITION: LEFT AND RIGHT HAND CONTINUITY AT A POINT** A function is left continuous at a point  $t_0$  if the right limit  $f(t_0^+)$ , defined as follows, exists

$$f(t_0^+) = \lim_{\delta \rightarrow 0} f(t_0 + \delta) \quad \delta > 0 \quad (2.40)$$

Similarly, a function is left continuous at a point  $t_0$  if the left limit  $f(t_0^-)$ , defined as follows, exists

$$f(t_0^-) = \lim_{\delta \rightarrow 0} f(t_0 - \delta) \quad \delta > 0 \quad (2.41)$$

**DEFINITION: PIECEWISE CONTINUITY AT A POINT** A function  $f$  is piecewise continuous at a point  $t_0$  if the left and right limits  $f(t_0^-)$  and  $f(t_0^+)$ , exist, that is,

$$\forall \varepsilon > 0 \quad \exists \delta_o > 0 \quad \text{s.t.} \quad 0 < \delta < \delta_o \Rightarrow |f(t_0 + \delta) - f(t_0^+)| < \varepsilon \quad (2.42)$$

$$\forall \varepsilon > 0 \quad \exists \delta_o > 0 \quad \text{s.t.} \quad 0 < \delta < \delta_o \Rightarrow |f(t_0^-) - f(t_0 - \delta)| < \varepsilon \quad (2.43)$$

and  $f(t_0) \in \{f(t_0^-), f(t_0^+)\}$ . Here,  $\text{s.t.}$  is an abbreviation for "such that." The last requirement excludes functions, such as

$$f(t) = \begin{cases} \infty & t = t_0 \\ k & t \neq t_0 \end{cases} \quad \text{or} \quad f(t) = \begin{cases} k_o & t = t_0 \\ k & t \neq t_0, k \neq k_o \end{cases} \quad (2.44)$$

from being piecewise continuous at  $t_0$ .

**DEFINITION: PIECEWISE CONTINUITY ON AN INTERVAL** A function  $f$  is piecewise continuous over an interval  $I$ , if it is piecewise continuous at all points in the interval  $I$ . For a closed interval  $[\alpha, \beta]$  right continuity is required at  $\alpha$  while left continuity is required at  $\beta$ .



---

sample content of Principles of Random Signal Analysis and Low Noise Design: The Power Spectral Density and its Applications

- [\*Stella Adler on America's Master Playwrights \(Eugene O'Neill, Clifford Odets, Tennessee Williams, Arthur Miller, Edward Albee, et al.\) for free\*](#)
- [download Horseshoe Crabs and Velvet Worms: The Story of the Animals and Plants That Time Has Left Behind online](#)
- [click Algorithmic Composition: Paradigms of Automated Music Generation for free](#)
- [\*\*read online Rethinking "Gnosticism": An Argument for Dismantling a Dubious Category bo\*\*](#)
  
- <http://dadhoc.com/lib/Power-Knowledge--Selected-Interviews-and-Other-Writings--1972-1977.pdf>
- <http://academialanguagebar.com/?ebooks/Let-Them-Eat-Vegan---200-Deliciously-Satisfying-Plant-Powered-Recipes-for-the-Whole-Family.pdf>
- <http://jaythebody.com/freebooks/Algorithmic-Composition--Paradigms-of-Automated-Music-Generation.pdf>
- <http://www.netc-bd.com/ebooks/Videohound-s-Golden-Movie-Retriever-2008--Videohound-s-Golden-Movie-Retriever-Series-.pdf>