BetterExplained Beta Notes (v0.5)

Welcome to the beta version of the ebook! My goal is to share the hard-won math insights that would have saved me hours of hand-wringing had I seen them earlier. The final version, which you will also receive, includes:

- Finalized layout & content
- Bonus Chapter: Understanding Euler's Theorem
- Afterward
- PowerPoint slides for all diagrams

Feedback

Information about the ebook is available at http://betterexplained.com/ebook. Comments and suggestions are welcome at kalid.azad@gmail.com.

Writing for BetterExplained is incredibly fulfilling, and I look forward to expanding its reach. Thanks for your support in this first step!

Math,
Better Explaine

by Kalid Azac

Let's keep it simple: I want you to enjoy math again. But the joy isn't in memorizing formulas through song, or doing examples with ice cream cones. It's in the exhilirating understanding of the idea behind the formula.

This book contains hard-won insights gathered over years of thinking, written in language that I wanted to see when first learning. Let's share the "aha!" moments!

Feedback

This book is meant to be fun, insightful and succinct. Feedback is welcome: kalid.azad@gmail.com, or on the website at http://betterexplained.com/ebook.

Why Buy the Book?

This book is a significant effort to arrange the content in a structured format for reading, printing, and presentations, along with brand-new discusisons of how the concepts fit together.

I'm just one learner hoping to improve math education; there's no corporation or foundation supporting this endeavor. These insights would have saved me countless hours of frustration if they'd been available when I was learning math. Please purchase the book t show your support.

How to use this book

This isn't a reference or workbook. It's a book about understanding ideas in a simple way.

I want concepts like e and i to be as natural to you as "a circle is round".

Here are a few practical ways to use the book:

- **Entertainment**: Yes, really. Math is fun when you see how ideas fit together and evolve. Did you know that negative numbers were only created in the 1700s, and were considered absurd? That imaginary numbers had the same fight when introduced? Me neither.
- **Study Supplement:** If you're a student, read this along with your textbook. Keep the analogies in your head as you do example problems to see how they fit into place.
- **Teaching Aid:** Teachers, parents, and other educators: feel free to incorporate the text, analogies or diagrams into your learning materials. Analogies and visualizations help enormously with puzzling concepts like imaginary numbers. The chapters inside incorporate the feedback of many thousands of readers.
- **Learn How to Learn:** The essays highlight my favorite learning method: get the history of an idea, formulate analogies, and cover examples using those analogies. Seeing how this technique works helps approach any subject, not just math.

The Legal Stuff

Math, Better Explained

1 Introduction

Our initial exposure to an idea shapes our intuition. And our intuition impacts how much we enjoy a subject. What do I mean?

Suppose we want to define a "cat":

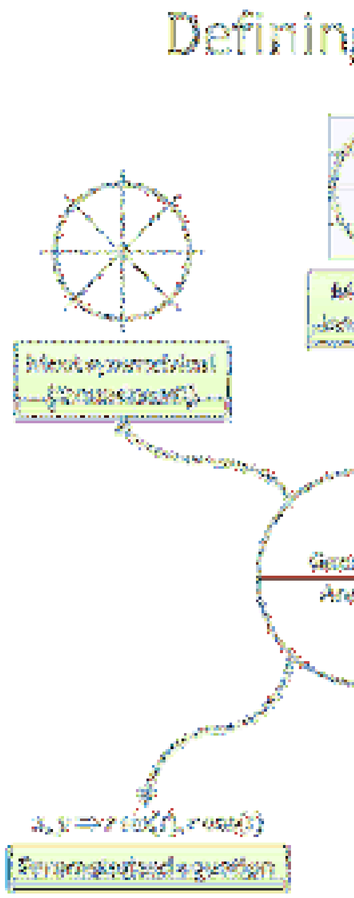
- **Caveman definition:** A furry animal with claws, teeth, a tail, 4 legs, that purrs when happy and hisses when angry…
- **Evolutionary definition:** Mammalian descendants of a certain species (F. catus), sharing certain characteristics…
- **Modern definition:** You call those definitions? Cats are animals sharing the following DNA: ACATACATACATACAT…

The modern definition is precise, sure. But is it the best? Is it what you'd teach a child learning the word? Does it give better insight into the "catness" of the animal? Not really. The modern definition is useful, but after getting an understanding of what a cat is. It shouldn't be our starting point.

Unfortunately, math understanding seems to follow the DNA pattern. We're taught the modern, rigorous definition and not the insights that led up to it. We're left with arcane formulas (DNA) but little understanding of what the idea is.

Let's approach ideas from a different angle. I imagine a circle: the center is the idea you're studying, and along the outside are the facts describing it. We start in one corner, with one fact or insight, and work our way around to develop our understanding. *Cats have common physical traits* leads to *Cats have a common ancestor* leads to *A species can be identified* *by certain portions of DNA*. Aha! I can see how the modern definition evolved from the caveman one.

But not all starting points are equal. The right perspective makes math click — and the mathematical "cavemen" who first found an idea often had an enlightening viewpoint. Let's learn how to build our intuition.



What is a Circle?

Time for a math example: How do you define a circle? There are seemingly countless definitions. Here's a few:

- The most symmetric 2-d shape possible
- The shape that gets the most area for the least perimeter (see the isoperimeter property)
- All points in a plane the same distance from a given point (drawn with a compass, or a pencil on a string)
- The points (x,y) in the equation $x^2 + y^2 = r^2$ (analytic version of the geometric definition above)
- The points in the equation r* sin(t), r* cos(t), for all t (really analytic version)
- The shape whose tangent line is always perpendicu-

lar to the position vector (physical interpretation)

The list goes on, but here's the key: the facts all describe the same idea! It's like saying 1, one, uno, eins, "the solution to $2x + 3 = 5$" or "the number of noses on your face" — just different names for the idea of "unity".

But these initial descriptions are important — they shape our intuition. Because we see circles in the real world before the classroom, we understand their "roundness". No matter what fancy equation we see ($x^2 + y^2 = r^2$), we know deep inside that a circle is "round". If we graphed that equation and it appeared square, or lopsided, we'd know there was a mistake.

As children, we learn the "caveman" definition of a circle (a really round thing), which gives us a comfortable intuition. We can see that every point on our "round thing" is the same distance from the center. $x^2 + y^2 = r^2$ is the analytic way of expressing that fact (using the Pythagorean theorem for distance). We started in one corner, with our intuition, and worked our way around to the formal definition.

Other ideas aren't so lucky. Do we instinctively see the *growth* of e, or is it an abstract definition? Do we realize the *rotation* of i, or is it an artificial, useless idea?

A Strategy For Developing Insight

I still have to remind myself about the deeper meaning of e and i — which seems as absurd as "remembering" that a circle is round or what a cat looks like! It should be the natural insight we start with.

Missing the big picture drives me crazy: math is about *ideas* — formulas are just a way to express them. Once the central concept is clear, the equations snap into place. Here's a strategy that has helped me:

- **Step 1: Find the central theme of a math concept.** This can be difficult, but tr starting with its history. Where was the idea first used? What was the discoverer doing? This use may be different from our modern interpretation and application.
- **Step 2: Explain a property/fact using the theme.** Use the theme to make an analogy to the formal definition. If you're lucky, you can translate the math equation ($x^2 + y^2 = r^2$) into a plain-english statement ("All points the same distance from the center").
- **Step 3: Explore related properties using the same theme.** Once you have ar analogy or interpretation that works, see if it applies to other properties. Sometimes it will, sometimes it won't (and you'll need a new insight), but you'd be surprised what you can discover.

Let's try it out.

A Real Example: Understanding e

Understanding the number e has been a major battle. e appears all of science, and has numerous definitions, yet rarely clicks in a natural way. Let's build some insight around this idea. The following section will have several equations, *which are simply ways to describe ideas*. Even if the equation is gibberish, there's a plain-english idea behind it.

Here's a few popular definitions of e: [TODO: refactor this section to be more book-friendly

The first step is to find a theme. Looking at e's history, it seems it has something to do with growth or interest rates. e was discovered when performing business calculations (not abstract mathematical conjectures) so "interest" (growth) is a possible theme.

Let's look at the first definition, in the upper left. The key jump, for me, was to realize how much this looked like the formula for compound interest. In fact, it is the interest formula when you compound 100% interest for 1 unit of time, compounding as fast as possible.

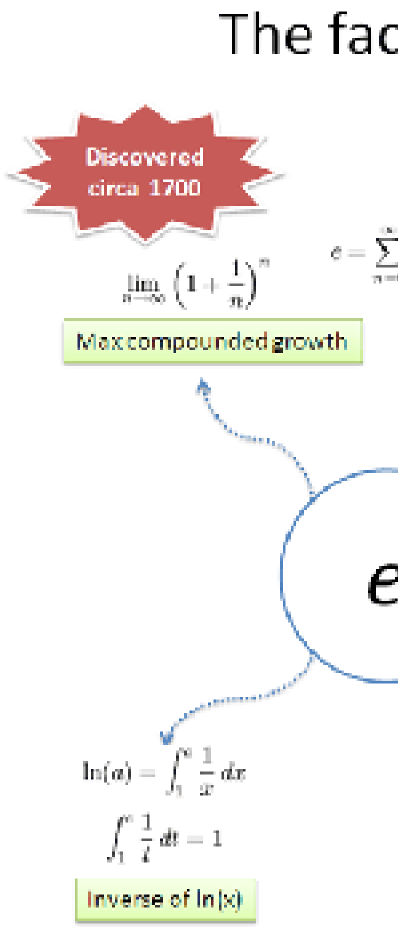
- **Definition 1:** Define e as 100% compound growth at the smallest increment possible.

The article on e describes this interpretation.

Let's look at the second definition: an infinite series of terms, getting smaller and smaller. What could this be?

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

After noodling this over using the theme of "interest" we see this definitions shows *the components of compound interest*. Now, insights don't come instantly — this insight might strike after brainstorming "What could 1 + 1 + 1/2 + 1/6 + ..." represent when talking



about growth?"

Well, the first term (1 1/0!, remembering that 0! is 1) is your principal, the original amount. The next term (1 1/1!) is the "direct" interest you earned 100% of 1. The next term (0.5 1/2!) is the amount of money your interest made ("2nd level interest"). The following term (.1666 1/3!) is your "3rd level interest" how much money your interest's interest earned!

Money earns money, which earns money, which earns money, and so on the sequence separates out these contributions (read the article on e to see how Mr. Blue, Mr. Green & Mr. Red grow independently). There's much more to say, but that's the "growth focused" understanding of that idea.

- **Definition 2:** Define e by the contributions each piece of interest makes

Neato.

Now to the 3rd, and shortest definition. What does it mean? Instead of thinking "derivative (which turns your brain into equation-crunching mode), think about what it means. The *feeling* of the equation. Make it your friend.

$$\frac{d}{dx} Blah = Blah$$

It's the calculus way of saying "Your rate of growth is equal to your current amount". Well, growing at your current amount would be a 100% interest rate, right? And by always growing it means you are *always calculating interest* — it's another way of describing continuously compound interest!

- **Definition 3:** Define e by as a function always growing by 100% of your current value

Nice — e is the number where you're always growing by exactly your current amount (100%), not 1% or 200%.

Time for the last definition — it's a tricky one. Here's my interpretation: Instead of describing how much you grew, why not say *how long it took*?

If you're at 1 and growing at 100%, it takes 1 unit of time to get from 1 to 2. But once you're at 2, and growing 100%, it means you're growing at 2 units per unit time! So it only takes 1/2 unit of time to go from 2 to 3. Going from 3 to 4 only takes 1/3 unit of time, and so on.

The time needed to grom from 1 to A is the time from 1 to 2, 2 to 3, 3 to 4... and so on, until you get to A. The first definition defines the natural log (ln) as shorthand for this "time to grow" computation.

ln(a) is simply the time to grow from 1 to a. We then say that "e" is the number that takes exactly 1 unit of time to grow to. Said another way, e is is the amount of growth after waiting exactly 1 unit of time!

- **Definition 4:** Define the time needed to grow continuously from 1 to as ln(a). e is the amount of growth you have after 1 unit of time.

Whablamo! These are four different ways to describe the mysterious e. Once we have the core idea ("e is about 100% continuous growth"), the crazy equations snap into place — it's "possible" to translate calculus into English. Math is about ideas!

What's the Moral?

In math class, we often start with the last, most complex idea. It's no wonder we're confused — we're showing DNA and expecting students to see the cat.

I've learned a few lessons from this approach, and it underlies how I understand and explai math:

- **Search for insights and apply them.** That first intuitive insight can help everything else snap into place. Start with a definition that makes sense and "walk around the circle" to find others.

- **Develop mental toughness.** Banging your head against an idea is no fun. If it doesn't click, come at it from different angles. There's another book, another article, another person who explains it in a way that makes sense to you.

- **It's ok to be visual.** We think of math as rigid and analytic — but visual interpreta tions are ok! Do what develops your understanding. Imaginary numbers were puzzling until their geometric interpretation came to light, decades after their initial discovery. Looking at equations all day didn't help mathematicians "get" what they were about.

Math becomes difficult and discouraging when we focus on definitions over understanding. Remember that the modern definition is the *most advanced* step of thought, not necessaril the starting point. Don't be afraid to approach a concept from a funny angle — figure out th plain-English sentence behind the equation. Happy math.

The Pythagorean theorem is a celebrity: if an equation can make it into the Simpsons, I'd say its well known.

$$a^2 - b^2 = c^2$$

But most of us think the formula only applies to triangles and geometry. Think again. The Pythagorean Theorem can be used with **any shape** and for **any formula that squares a number.**

Read on to see how this 2500-year-old idea can help us understand computer science, phys ics, even the value of Web 2.0 social networks.

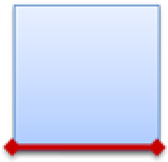
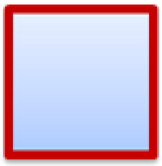
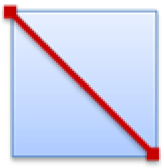
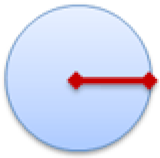
Understanding How Area Works

I love seeing old topics in a new light and discovering the depth there. For example, I realiz I didn't have a deep grasp of area until writing this article. Yes, we can rattle off equations, but do we really *understand* the nature of area? This fact may surprise you:

The area of any shape can be computed from any line segment squared. In a square, our "line segment" is usually a side, and the area is that side squared (side 5, area 25). In a circle, the line segment is often the radius, and the area is pi * r^2 (radius 5, area 25 pi). Easy enough.

We can pick any line segment and figure out area from it: every line segment has an "area factor" in this universal equation:

$$Area = Factor \cdot (line\ segment)^2$$

Shape	Line Segment	Area	Area Factor
Square 	Side [s]	s^2	1
Square 	Perimeter [p]	$1/16\ p^2$	1/16
Square 	Diagonal [d]	$1/2\ d^2$	1/2
Circle 	Radius [r]	$pi\ r^2$	pi (3.14159...)

For example, look at the diagonal of a square ("d"). A regular side is d/sqrt(2), so the area becomes $1/2\ d^2$. Our "area contant" is 1/2 in this case, if we want to use the diagonal as our line segment to be squared.

Now, use the entire perimeter ("p") as the line segment. A side is p/4, so the area is $p^2/16$. The area factor is 1/16 if we want to use p^2.

Can we pick any line segment?

You bet. There is always some relationship between the "traditional" line segment (the side of a square), and the one you pick (the perimeter, which happens to be 4 times a side). Sinc we can convert between the "traditional" and "new" segment, it doesn't matter which one w use -- there'll just be a different area factor when we multiply it out.

Can we pick any shape?

Sort of. A given area formula works for all similar shapes, where "similar" means "zoomed versions of each other". For example:

- All squares are similar (area always s^2)
- All circles are similar, too (area always pi * r^2)
- Triangles are not similar: Some are fat and others skinny -- every "type" of triangle has its own area factor based on the line segment you are using. Change the shape o the triange and the equation changes.

Yes, every triangle follows the rule "area = 1/2 base * height". But the relationship between base and height depends on the type of triangle (base = 2 * height, base = 3 * height, etc.), s even then the area factor will be different.

Why do we need similar shapes to keep the same area equation? Intuitively, when you zoom (scale) a shape, you're changing the absolute size but not the relative ratios within the shape. A square, no matter how zoomed, has a perimeter = 4 * side.

Because the "area factor" is based on ratios inside the shape, any shapes with the same "ratios" will follow the same formula. It's a bit like saying everyone's armspan is about equal to their height. No matter if you're a NBA basketball player or child, the equation holds because it's all relative. (This intuitive argument may not satisfy a mathematical mind -- in that case, take up your concerns with Euclid).

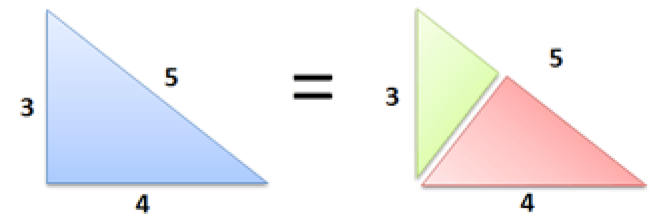
I hope these high-level concepts make sense:

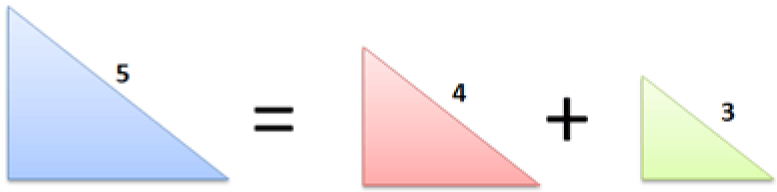
- **Area can be be found from any line segment squared**, not just the "side" or "radius"
- Each line segment has a different "area factor"
- The same area equation works for similar shapes

Intuitive Look at The Pythagorean Theorem

We can all agree the Pythagorean Theorem is true (here's 75 proofs). But most proofs offer a mechanical understanding: re-arrange the shapes, and voila, the equation holds. But is it really clear, intuitively, that it must be $a^2 + b^2 = c^2$ and not $2a^2 + b^2 = c^2$? No? Well, let's build some intuition.

There's one killer concept we need: **Any right triangle can be split into two similar right triangles.**





Cool, huh? Drawing a perpendicular line through the point splits a right triangle into two smaller ones. Geometry lovers, try the proof yourself: use angle-angle-angle similarity.

This diagram also makes something very clear:

- Area (Big) = Area (Medium) + Area (Small)

Makes sense, right? The smaller trianges were cut from the big one, so the areas must add up. And the kicker: **because the triangles are similar, they have the same area equation.**

Let's call the long side c (5), the middle side b (4), and the small side a (3). Our area equation for these triangles is:

$Area = F * hypotenuse^2$

where F is some area factor (6/25 or .24 in this case; the exact number doesn't matter). Now let's play with the equation:

$Area(Big) = Area(Medium) + Area(Small)$

$Fc^2 = Fb^2 + Fa^2$

Divide by F on both sides and you get:

$c^2 = b^2 + a^2$

Which is our famous theorem! You knew it was true, but now you know why:

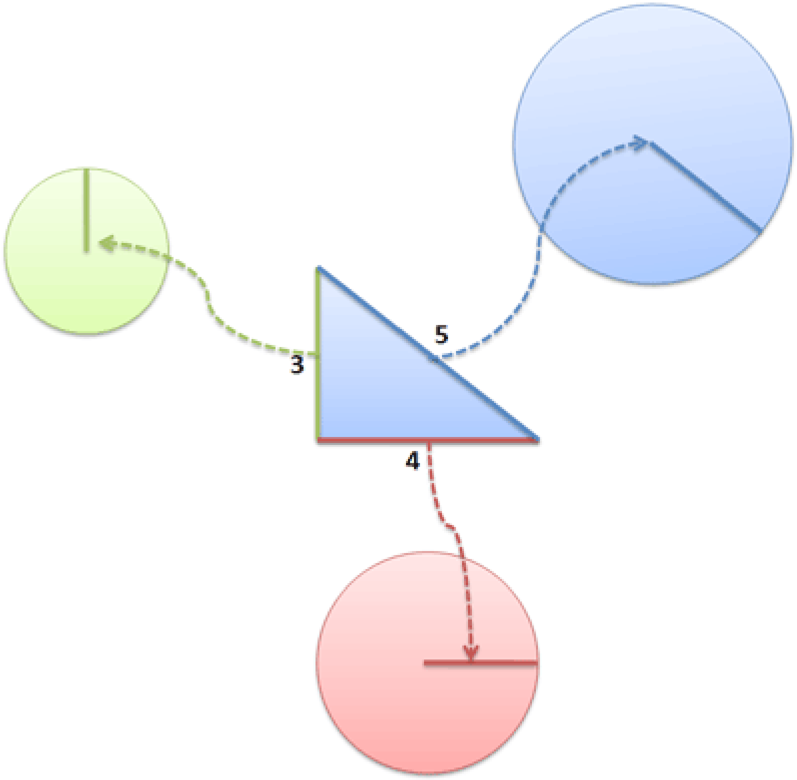
- A triangle can be split into two smaller, similar ones
- Since the areas must add up, the squared hypotenuses (which determine area) must add up as well.

This takes a bit of time to see, but I hope the result is clear. How could the small triangles not add to the larger one?

Actually, it turns out the Pythagorean Theorem depends on the assumptions of Euclidean geometry and doesn't work on spheres or globes, for example. But we'll save that discussion for another time.

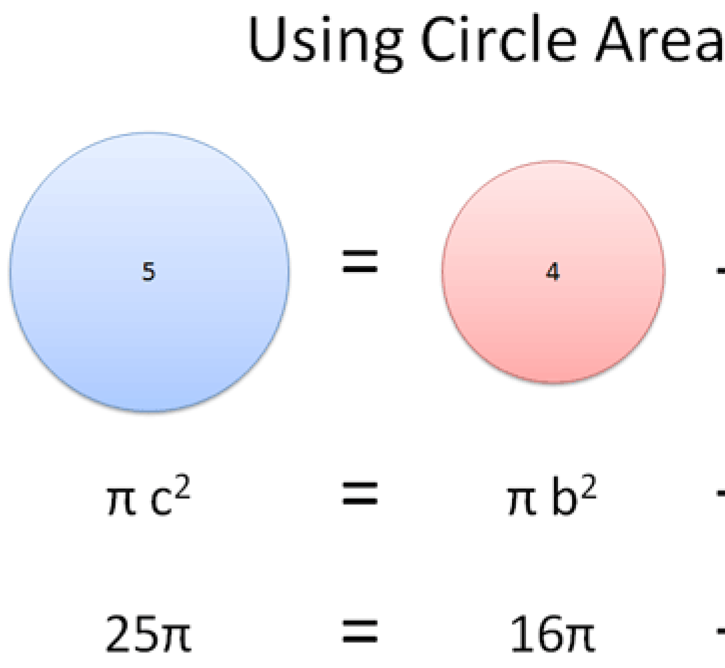
Useful Application: Try Any Shape

We used triangles in our diagram, the simplest 2-D shape. But the line segment can belong to any shape. Take circles, for example:



Now what happens when we add them together?

You guessed it: Circle of radius 5 = Circle of radius 4 + Circle of radius 3.



Pretty wild, eh? We can multiply the Pythagorean Theorem by our area factor (pi, in this case) and come up with a relationship for any shape.

Remember, the line segment can be **any portion of the shape**. We could have picked the circle's radius, diameter, or circumference -- there would be a different area factor, but the 3-4-5 relationship would still hold.

So, whether you're adding up pizzas or Richard Nixon masks, the Pythagorean theorem helps you relate the areas of any similar shapes. Now that's something they didn't teach you in grade school.

Useful Application: Conservation of Squares

The Pythagorean Theorem applies to any equation that has a square. The triangle-splitting means you can split any amount (c^2) into two smaller amounts ($a^2 + b^2$) based on the sides o a right triangle. In reality, the "length" of a side can be distance, energy, work, time, or even people in a social network:

Social Networks.

Metcalfe's Law (if you believe it) says the value of a network is about n^2 (the number of relationships). In terms of value,

- Network of 50M = Network of 40M + Network of 30M.

Pretty amazing -- the 2nd and 3rd networks have 70M people total, but they aren't a coherent whole. The network with 50 million people is as valuable as the others combined.

Computer Science

Some programs with n inputs take n^2 time to run (bubble sort, for example). In terms of processing time:

- 50 inputs = 40 inputs + 30 inputs

Pretty interesting. 70 elements spread among two groups can be sorted as fast as 50 items i one group. (Yeah, there may be constant overhead/start up time, just work with me here).

Given this relationship, it makes sense to partition elements into separate groups and then sort the subgroups. Indeed, that's the approach used in quicksort, one of the best general-purpose sorting methods. The Pythagorean theorem helps show how sorting 50 combined elements can be as slow as sorting 30 and 40 separate ones.

Surface Area

The surface area of a sphere is 4 pi r^2. So, in terms of surface area of spheres:

- Area of radius 50 = area of radius 40 + area of radius 30

We don't often have spheres lying around, but boat hulls may have the same relationship (they're like deformed spheres, right?). Assuming the boats are similarly shaped, the paint needed to coat one 50 foot yacht could instead paint a 40 and 30-footer. Yowza.

Physics

If you remember your old physics classes, the kinetic energy of an object with mass m and velocity v is **1/2 m v^2**. In terms of energy,

- Energy at 500 mph = Energy at 400 mph + Energy at 300 mph

With the energy used to accelerate one bullet to 500 mph, we could accelerate two others to 400 and 300 mph.

Try Any Number

You can use any set of numbers that make a right triangle. For example, enter a total amount (50) and one subportion (30), and the remainder will appear below: [TODO: link to instacalc].

Suppose you want to see if a large pizza (16 inches) is bigger than two mediums (12 inches). Plug in 16 for C, and 12 for A. It looks like the large pizza can be split into a 12-inch and 10.5 inch pizza, so two-mediums are in fact bigger.

Enjoy Your New Insight