Tyn Myint-U
Lokenath Debnath

Linear Partial Differential Equations for Scientists and Engineers

Fourth Edition

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To the Memory of

U and Mrs. Hla Din

U and Mrs. Thant

Tyn Myint-U

In Loving Memory of

My Mother and Father

Lokenath Debnath

“True Laws of Nature cannot be linear.”

“The search for truth is more precious than its possession.”

“Everything should be made as simple as possible, but not a bit simpler.”

Albert Einstein

“No human investigation can be called real science if it cannot be demonstrated mathematically.”

Leonardo Da Vinci

“First causes are not known to us, but they are subjected to simple and constant laws that can be studied by observation and whose study is the goal of Natural Philosophy ... Heat penetrates, as does gravity, all the substances of the universe; its rays occupy all regions of space. The aim of our work is to expose the mathematical laws that this element follows ... The differential equations for the propagation of heat express the most general conditions and reduce physical questions to problems in pure Analysis that is properly the object of the theory.”

James Clerk Maxwell

“One of the properties inherent in mathematics is that any real progress is accompanied by the discovery and development of new methods and simplifications of previous procedures ... The unified character of mathematics lies in its very nature; indeed, mathematics is the foundation of all exact natural sciences.”

David Hilbert

“ ... partial differential equations are the basis of all physical theorems. In the theory of sound in gases, liquid and solids, in the investigations of elasticity, in optics, everywhere partial differential equations formulate basic laws of nature which can be checked against experiments.”

Bernhard Riemann

“The effective numerical treatment of partial differential equations is not a handicraft, but an art.”

Folklore

“The advantage of the principle of least action is that in one and the same equation it relates the quantities that are immediately relevant not only to mechanics but also to electrodynamics and thermodynamics; these quantities are space, time and potential.”

Max Planck

“The thorough study of nature is the most ground for mathematical discoveries.”

“The equations for the flow of heat as well as those for the oscillations of acoustic bodies and of fluids belong to an area of analysis which has recently been opened, and which is worth examining in the greatest detail.”

Joseph Fourier

“Of all the mathematical disciplines, the theory of differential equation is the most important. All branches of physics pose problems which can be reduced to the integration of differential equations. More generally, the way of explaining all natural phenomena which depend on time is given by the theory of differential equations.”

Sophus Lie

“Differential equations form the basis for the scientific view of the world.”

V.I. Arnold

“What we know is not much. What we do not know is immense.”

“The algebraic analysis soon makes us forget the main object [of our research] by focusing our attention on abstract combinations and it is only at the end that we return to the original objective. But in abandoning oneself to the operations of analysis, one is led to the generality of this method and the inestimable advantage of transforming the reasoning by mechanical procedures to results often inaccessible by geometry ... No other language has the capacity for the elegance that arises from a long sequence of expressions linked one to the other and all stemming from one fundamental idea.”

“It is India that gave us the ingenious method of expressing all numbers by ten symbols, each symbol receiving a value of position, as well as an absolute value. We shall appreciate the grandeur of the achievement when we remember that it escaped the genius of Archimedes and Appolonius.”

P.S. Laplace

“The mathematician’s best work is art, a high perfect art, as daring as the most secret dreams of imagination, clear and limpid. Mathematical genius and artistic genius touch one another.”

Gȯsta Mittag-Leffler

Contents

Preface to the Fourth Edition

"A teacher can never truly teach unless he is still learning himself. A lamp can never light another lamp unless it continues to burn its own flame. The teacher who has come to the end of his subject, who has no living traffic with his knowledge but merely repeats his lessons to his students, can only load their minds; he cannot quicken them."

Rabindranath Tagore
An Indian Poet
1913 Nobel Prize Winner for Literature

The previous three editions of our book were very well received and used as a senior undergraduate or graduate-level text and research reference in the United States and abroad for many years. We received many comments and suggestions from many students, faculty and researchers around the world. These comments and criticisms have been very helpful, beneficial, and encouraging. This fourth edition is the result of the input.

Another reason for adding this fourth edition to the literature is the fact that there have been major discoveries of new ideas, results and methods for the solution of linear and nonlinear partial differential equations in the second half of the twentieth century. It is becoming even more desirable for mathematicians, scientists and engineers to pursue study and research on these topics. So what has changed, and will continue to change is the nature of the topics that are of interest in mathematics, applied mathematics, physics and engineering, the evolution of books such is this one is a history of these shifting concerns.

This new and revised edition preserves the basic content and style of the third edition published in 1989. As with the previous editions, this book has been revised primarily as a comprehensive text for senior undergraduates or beginning graduate students and a research reference for professionals in mathematics, science and engineering, and other applied sciences. The main goal of the book is to develop required analytical skills on the part of the reader, rather than to focus on the importance of more abstract formulation, with full mathematical rigor. Indeed, our major emphasis is to provide an accessible working knowledge of the analytical and numerical methods with proofs required in mathematics, applied mathematics, physics, and engineering. The revised edition was greatly influenced by the statements that Lord Rayleigh and Richard Feynman made as follows:

"In the mathematical investigation I have usually employed such methods as present themselves naturally to a physicist. The pure mathematician will complain, and (it must be confessed) sometimes with justice, of deficient rigor. But to this question there are two sides. For, however important it may be to maintain a uniformly high standard in pure mathematics, the physicist may occasionally do well to rest content with arguments, which are fairly satisfactory and conclusive from his point of view. To his mind, exercised in a different order of ideas, the more severe procedure of the pure mathematician may appear not more but less demonstrative. And further, in many cases of difficulty to insist upon highest standard would mean the exclusion of the subject altogether in view of the space that would be required."

Lord Rayleigh

"... However, the emphasis should be somewhat more on how to do the mathematics quickly and easily, and what formulas are true, rather than the mathematicians' interest in methods of rigorous proof."

Richard P. Feynman

We have made many additions and changes in order to modernize the contents and to improve the clarity of the previous edition. We have also taken advantage of this new edition to correct typographical errors, and to update the bibliography, to include additional topics, examples of applications, exercises, comments and observations, and in some cases, to entirely rewrite and reorganize many sections. This is plenty of material in the book for a year-long course. Some of the material need not be covered in a course work and can be left for the readers to study on their own in order to prepare them for further study and research. This edition contains a collection of over 900 worked examples and exercises with answers and hints to selected exercises. Some of the major changes and additions include the following:

1. Chapter 1 on Introduction has been completely revised and a new section on historical comments was added to provide information about the historical developments of the subject. These changes have been made to provide the reader to see the direction in which the subject has developed and find those contributed to its developments.
2. A new Chapter 2 on first-order, quasi-linear, and linear partial differential equations, and method of characteristics has been added with many new examples and exercises.

3. Two sections on conservation laws, Burgers' equation, the Schrödinger and the Korteweg-de Vries equations have been included in Chapter 3.
4. Chapter 6 on Fourier series and integrals with applications has been completely revised and new material added, including a proof of the pointwise convergence theorem.
5. A new section on fractional partial differential equations has been added to Chapter 12 with many new examples of applications.
6. A new section on the Lax pair and the Zakharov and Shabat Scheme has been added to Chapter 13 to modernize its contents.
7. Some sections of Chapter 14 have been revised and a new short section on the finite element method has been added to this chapter.
8. A new Chapter 15 on tables of integral transforms has been added in order to make the book self-contained.
9. The whole section on Answers and Hints to Selected Exercises has been expanded to provide additional help to students. All figures have been redrawn and many new figures have been added for a clear understanding of physical explanations.
10. An Appendix on special functions and their properties has been expanded.

Some of the highlights in this edition include the following:

- The book offers a detailed and clear explanation of every concept and method that is introduced, accompanied by carefully selected worked examples, with special emphasis given to those topics in which students experience difficulty.
- A wide variety of modern examples of applications has been selected from areas of integral and ordinary differential equations, generalized functions and partial differential equations, quantum mechanics, fluid dynamics and solid mechanics, calculus of variations, linear and nonlinear stability analysis.
- The book is organized with sufficient flexibility to enable instructors to select chapters appropriate for courses of differing lengths, emphases, and levels of difficulty.
- A wide spectrum of exercises has been carefully chosen and included at the end of each chapter so the reader may further develop both rigorous skills in the theory and applications of partial differential equations and a deeper insight into the subject.
- Many new research papers and standard books have been added to the bibliography to stimulate new interest in future study and research. Index of the book has also been completely revised in order to include a wide variety of topics.
- The book provides information that puts the reader at the forefront of current research.

With the improvements and many challenging worked-out problems and exercises, we hope this edition will continue to be a useful textbook for students as well as a research reference for professionals in mathematics, applied mathematics, physics and engineering.

It is our pleasure to express our grateful thanks to many friends, colleagues, and students around the world who offered their suggestions and help at various stages of the preparation of the book. We offer special thanks to Dr. Andras Balogh, Mr. Kanadpriya Basu, and Dr. Dambaru Bhatta for drawing all figures, and to Mrs. Veronica Martinez for typing the manuscript with constant changes and revisions. In spite of the best efforts of everyone involved, some typographical errors doubtless remain. Finally, we wish to express our special thanks to Tom Grasso and the staff of Birkhäuser Boston for their help and cooperation.

Tyn Myint-U

Lokenath Debnath

Preface to the Third Edition

The theory of partial differential equations has long been one of the most important fields in mathematics. This is essentially due to the frequent occurrence and the wide range of applications of partial differential equations in many branches of physics, engineering, and other sciences. With much interest and great demand for theory and applications in diverse areas of science and engineering, several excellent books on PDEs have been published. This book is written to present an approach based mainly on the mathematics, physics, and engineering problems and their solutions, and also to construct a course appropriate for all students of mathematical, physical, and engineering sciences. Our primary objective, therefore, is not concerned with an elegant exposition of general theory, but rather to provide students with the fundamental concepts, the underlying principles, a wide range of applications, and various methods of solution of partial differential equations.

This book, a revised and expanded version of the second edition published in 1980, was written for a one-semester course in the theory and applications of partial differential equations. It has been used by advanced undergraduate or beginning graduate students in applied mathematics, physics, engineering, and other applied sciences. The prerequisite for its study is a standard calculus sequence with elementary ordinary differential equations. This revised edition is in part based on lectures given by Tyn Myint-U at Manhattan College and by Lokenath Debnath at the University of Central Florida. This revision preserves the basic content and style of the earlier editions, which were written by Tyn Myint-U alone. However, the authors have made some major additions and changes in this third edition in order to modernize the contents and to improve clarity. Two new chapters added are on nonlinear PDEs, and on numerical and approximation methods. New material emphasizing applications has been inserted. New examples and exercises have been provided. Many physical interpretations of mathematical solutions have been added. Also, the authors have improved the exposition by reorganizing some material and by making examples, exercises, and applications more prominent in the text. These additions and changes have been made with the student uppermost in mind.

The first chapter gives an introduction to partial differential equations. The second chapter deals with the mathematical models representing physical and engineering problems that yield the three basic types of PDEs. Included are only important equations of most common interest in physics and engineering. The third chapter constitutes an account of the classification of linear PDEs of second order in two independent variables into hyperbolic, parabolic, and elliptic types and, in addition, illustrates the determination of the general solution for a class of relatively simple equations.

Cauchy's problem, the Goursat problem, and the initial boundary-value problems involving hyperbolic equations of the second order are presented in Chapter 4. Special attention is given to the physical significance of solutions and the methods of solution of the wave equation in Cartesian, spherical polar, and cylindrical polar coordinates. The fifth chapter contains a fuller treatment of Fourier series and integrals essential for the study of PDEs. Also included are proofs of several important theorems concerning Fourier series and integrals.

Separation of variables is one of the simplest methods, and the most widely used method, for solving PDEs. The basic concept and separability conditions necessary for its application are discussed in the sixth chapter. This is followed by some well-known problems of applied mathematics, mathematical physics, and engineering sciences along with a detailed analysis of each problem. Special emphasis is also given to the existence and uniqueness of the solutions and to the fundamental similarities and differences in the properties of the solutions to the various PDEs. In Chapter 7, self-adjoint eigenvalue problems are treated in depth, building on their introduction in the preceding chapter. In addition, Green's function and its applications to eigenvalue problems and boundary-value problems for ordinary differential equations are presented. Following the general theory of eigenvalues and eigenfunctions, the most common special functions, including the Bessel, Legendre, and Hermite functions, are discussed as examples of the major role of special functions in the physical and engineering sciences. Applications to heat conduction problems and the Schrödinger equation for the linear harmonic oscillator are also included.

Boundary-value problems and the maximum principle are described in Chapter 8, and emphasis is placed on the existence, uniqueness, and well-posedness of solutions. Higher-dimensional boundary-value problems and the method of eigenfunction expansion are treated in the ninth chapter, which also includes several applications to the vibrating membrane, waves in three dimensions, heat conduction in a rectangular volume, the three-dimensional Schrödinger equation in a central field of force, and the hydrogen atom. Chapter 10 deals with the basic concepts and construction of Green's function and its application to boundary-value problems.

Chapter 11 provides an introduction to the use of integral transform methods and their applications to numerous problems in applied mathematics, mathematical physics, and engineering sciences. The fundamental properties and the techniques of Fourier, Laplace, Hankel, and Mellin transforms are discussed in some detail. Applications to problems concerning heat flows, fluid flows, elastic waves, current and potential electric transmission lines are included in this chapter.

Chapters 12 and 13 are entirely new. First-order and second-order nonlinear PDEs are covered in Chapter 12. Most of the contents of this chapter have been developed during the last twenty-five years. Several new nonlinear PDEs including the one-dimensional nonlinear wave equation, Whitham's equation, Burgers' equation, the Korteweg–de Vries equation, and the nonlinear Schrödinger equation are solved. The solutions of these equations are then discussed with physical significance. Special emphasis is given to the fundamental similarities and differences in the properties of the solutions to the corresponding linear and nonlinear equations under consideration.

The final chapter is devoted to the major numerical and approximation methods for finding solutions of PDEs. A fairly detailed treatment of explicit and implicit finite difference methods is given with applications The variational method and the Euler–Lagrange equations are described with many applications. Also included are the Rayleigh–Ritz, the Galerkin, and the Kantorovich methods of approximation with many illustrations and applications.

This new edition contains almost four hundred examples and exercises, which are either directly associated with applications or phrased in terms of the physical and engineering contexts in which they arise. The exercises truly complement the text, and answers to most exercises are provided at the end of the book. The Appendix has been expanded to include some basic properties of the Gamma function and the tables of Fourier, Laplace, and Hankel transforms. For students wishing to know more about the subject or to have further insight into the subject matter, important references are listed in the Bibliography.

The chapters on mathematical models, Fourier series and integrals, and eigenvalue problems are self-contained, so these chapters can be omitted for those students who have prior knowledge of the subject.

An attempt has been made to present a clear and concise exposition of the mathematics used in analyzing a variety of problems. With this in mind, the chapters are carefully organized to enable students to view the material in an orderly perspective. For example, the results and theorems in the chapters on Fourier series and integrals and on eigenvalue problems are explicitly mentioned, whenever necessary, to avoid confusion with their use in the development of PDEs. A wide range of problems subject to various boundary conditions has been included to improve the student's understanding.

In this third edition, specific changes and additions include the following:

1. Chapter 2 on mathematical models has been revised by adding a list of the most common linear PDEs in applied mathematics, mathematical physics, and engineering science.
2. The chapter on the Cauchy problem has been expanded by including the wave equations in spherical and cylindrical polar coordinates. Examples and exercises on these wave equations and the energy equation have been added.
3. Eigenvalue problems have been revised with an emphasis on Green's functions and applications. A section on the Schrödinger equation for the linear harmonic oscillator has been added. Higher-dimensional boundary-value problems with an emphasis on applications, and a section on the hydrogen atom and on the three-dimensional Schrödinger equation in a central field of force have been added to Chapter 9.
4. Chapter 11 has been extensively reorganized and revised in order to include Hankel and Mellin transforms and their applications, and has new sections on the asymptotic approximation method and the finite Hankel transform with applications. Many new examples and exercises, some new material with applications, and physical interpretations of mathematical solutions have also been included.
5. A new chapter on nonlinear PDEs of current interest and their applications has been added with considerable emphasis on the fundamental similarities and the distinguishing differences in the properties of the solutions to the nonlinear and corresponding linear equations.
6. Chapter 13 is also new. It contains a fairly detailed treatment of explicit and implicit finite difference methods with their stability analysis. A large section on the variational methods and the Euler–Lagrange equations has been included with many applications. Also included are the Rayleigh–Ritz, the Galerkin, and the Kantorovich methods of approximation with illustrations and applications.
7. Many new applications, examples, and exercises have been added to deepen the reader's understanding. Expanded versions of the tables of Fourier, Laplace, and Hankel transforms are included. The bibliography has been updated with more recent and important references.

As a text on partial differential equations for students in applied mathematics, physics, engineering, and applied sciences, this edition provides the student with the art of combining mathematics with intuitive and physical thinking to develop the most effective approach to solving problems.

In preparing this edition, the authors wish to express their sincere thanks to those who have read the manuscript and offered many valuable suggestions and comments. The authors also wish to express their thanks to the editor and the staff of Elsevier–North Holland, Inc. for their kind help and cooperation.

Tyn Myint-U
Lokenath Debnath

1

Introduction

"If you wish to foresee the future of mathematics, our proper course is to study the history and present condition of the science."

Henri Poincaré

"However varied may be the imagination of man, nature is a thousand times richer, ... Each of the theories of physics ... presents (partial differential) equations under a new aspect ... without the theories, we should not know partial differential equations."

Henri Poincaré

1.1 Brief Historical Comments

Historically, partial differential equations originated from the study of surfaces in geometry and a wide variety of problems in mechanics. During the second half of the nineteenth century, a large number of famous mathematicians became actively involved in the investigation of numerous problems presented by partial differential equations. The primary reason for this research was that partial differential equations both express many fundamental laws of nature and frequently arise in the mathematical analysis of diverse problems in science and engineering.

The next phase of the development of linear partial differential equations was characterized by efforts to develop the general theory and various methods of solution of linear equations. In fact, partial differential equations have been found to be essential to the theory of surfaces on the one hand and to the solution of physical problems on the other. These two areas of mathematics can be seen as linked by the bridge of the calculus of variations. With the discovery of the basic concepts and properties of distributions, the modern theory of linear partial differential equations is now well established. The subject plays a central role in modern mathematics, especially in physics, geometry, and analysis.

Almost all physical phenomena obey mathematical laws that can be formulated by differential equations. This striking fact was first discovered by Isaac Newton (1642–1727) when he formulated the laws of mechanics and applied them to describe the motion of the planets. During the three centuries since Newton's fundamental discoveries, many partial differential equations that govern physical, chemical, and biological phenomena have been found and successfully solved by numerous methods. These equations include Euler's equations for the dynamics of rigid bodies and for the motion of an ideal fluid, Lagrange's equations of motion, Hamilton's equations of motion in analytical mechanics, Fourier's equation for the diffusion of heat, Cauchy's equation of motion and Navier's equation of motion in elasticity, the Navier–Stokes equations for the motion of viscous fluids, the Cauchy–Riemann equations in complex function theory, the Cauchy–Green equations for the static and dynamic behavior of elastic solids, Kirchhoff's equations for electrical circuits, Maxwell's equations for electromagnetic fields, and the Schrödinger equation and the Dirac equation in quantum mechanics. This is only a sampling, and the recent mathematical and scientific literature reveals an almost unlimited number of differential equations that have been discovered to model physical, chemical and biological systems and processes.

From the very beginning of the study, considerable attention has been given to the geometric approach to the solution of differential equations. The fact that families of curves and surfaces can be defined by a differential equation means that the equation can be studied geometrically in terms of these curves and surfaces. The curves involved, known as *characteristic curves*, are very useful in determining whether it is or is not possible to find a surface containing a given curve and satisfying a given differential equation. This geometric approach to differential equations was begun by Joseph-Louis Lagrange (1736–1813) and Gaspard Monge (1746–1818). Indeed, Monge first introduced the ideas of characteristic surfaces and characteristic cones (or Monge cones). He also did some work on second-order linear, homogeneous partial differential equations.

The study of first-order partial differential equations began to receive some serious attention as early as 1739, when Alex-Claude Clairaut (1713–1765) encountered these equations in his work on the shape of the earth. On the other hand, in the 1770s Lagrange first initiated a systematic study of the first-order nonlinear partial differential equations in the form

$$f(x, y, u, u_x, u_y) = 0, \tag{1.1.1}$$

where $u = u(x, y)$ is a function of two independent variables.

Motivated by research on gravitational effects on bodies of different shapes and mass distributions, another major impetus for work in partial differential equations originated from potential theory. Perhaps the most important partial differential equation in applied mathematics is the *potential equation*, also known as the *Laplace equation* $u_{xx} + u_{yy} = 0$, where subscripts denote partial derivatives. This equation arose in steady state heat conduction problems involving homogeneous solids. James Clerk Maxwell (1831–1879) also gave a new initiative to potential theory through his famous equations, known as *Maxwell's equations* for electromagnetic fields.

Lagrange developed analytical mechanics as the application of partial differential equations to the motion of rigid bodies. He also described the geometrical content of a first-order partial differential equation and developed the method of characteristics for finding the general solution of quasi-linear equations. At the same time, the specific solution of physical interest was obtained by formulating an *initial-value problem* (or a *Cauchy Problem*) that satisfies certain supplementary conditions. The solution of an initial-value problem still plays an important role in applied mathematics, science and engineering. The fundamental role of characteristics was soon recognized in the study of quasi-linear and nonlinear partial differential equations. Physically, the first-order, quasi-linear equations often represent conservation laws which describe the conservation of some physical quantities of a system.

In its early stages of development, the theory of second-order linear partial differential equations was concentrated on applications to mechanics and physics. All such equations can be classified into three basic categories: the wave equation, the heat equation, and the Laplace equation (or potential equation). Thus, a study of these three different kinds of equations yields much information about more general second-order linear partial differential equations. Jean d'Alembert (1717–1783) first derived the one-dimensional wave equation for vibration of an elastic string and solved this equation in 1746. His solution is now known as the *d'Alembert solution*. The wave equation is one of the oldest equations in mathematical physics. Some form of this equation, or its various generalizations, almost inevitably arises in any mathematical analysis of phenomena involving the propagation of waves in a continuous medium. In fact, the studies of water waves, acoustic waves, elastic waves in solids, and electromagnetic waves are all based on this equation. A technique known as the *method of separation of variables* is perhaps one of the oldest systematic methods for solving partial differential equations including the wave equation. The wave equation and its methods of solution attracted the attention of many famous mathematicians including Leonhard Euler (1707–1783), James Bernoulli (1667–1748), Daniel Bernoulli (1700–1782), J.L. Lagrange (1736–1813), and Jacques Hadamard (1865–1963). They discovered solutions in several different forms, and the merit of their solutions and relations among these solutions were argued in a series of papers extending over more than twenty-five years; most concerned the nature of the kinds of functions that can be represented by trigonometric (or Fourier) series. These controversial problems were finally resolved during the nineteenth century.

It was Joseph Fourier (1768–1830) who made the first major step toward developing a general method of solutions of the equation describing the conduction of heat in a solid body in the early 1800s. Although Fourier is most celebrated for his work on the conduction of heat, the mathematical methods involved, particularly trigonometric series, are important and very useful in many other situations. He created a coherent mathematical method by which the different components of an equation and its solution in series were neatly identified with the different aspects of the physical solution being analyzed. In spite of the striking success of Fourier analysis as one of the most useful mathematical methods, J.L. Lagrange and S.D. Poisson (1781–1840) hardly recognized Fourier's work because of its lack of rigor. Nonetheless, Fourier was eventually recognized for his pioneering work after publication of his monumental treatise entitled *La Théorie Auatytique de la Chaleur* in 1822.

It is generally believed that the concept of an integral transform originated from the Integral Theorem as stated by Fourier in his 1822 treatise. It was the work of Augustin Cauchy (1789–1857) that contained the exponential form of the Fourier Integral Theorem as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \left[\int_{-\infty}^{\infty} e^{-ik\xi} f(\xi)\, d\xi \right] dk. \tag{1.1.2}$$

This theorem has been expressed in several slightly different forms to better adapt it for particular applications. It has been recognized, almost from the start, however, that the form which best combines mathematical simplicity and complete generality makes use of the exponential oscillating function $\exp(ikx)$. Indeed, the Fourier integral formula (1.1.2) is regarded as one of the most fundamental results of modern mathematical analysis, and it has widespread physical and engineering applications. The generality and importance of the theorem is well expressed by Kelvin and Tait who said: " ... Fourier's Theorem, which is not only one of the most beautiful results of modern analysis, but may be said to furnish an indispensable instrument in the treatment of nearly every recondite question in modern physics. To mention only sonorous vibrations, the propagation of electric signals along a telegraph wire, and the conduction of heat by the earth's crust, as subjects in their generality intractable without it, is to give but a feeble idea of its importance." This integral formula (1.1.2) is usually used to define the classical Fourier transform of a function and the inverse Fourier transform. No doubt, the scientific achievements of Joseph Fourier have not only provided the fundamental basis for the study of heat equation, Fourier series, and Fourier integrals, but for the modern developments of the theory and applications of the partial differential equations.

One of the most important of all the partial differential equations involved in applied mathematics and mathematical physics is that associated with the name of Pierre-Simon Laplace (1749–1827). This equation was first discovered by Laplace while he was involved in an extensive study of