Undergraduate Texts in Mathematics

UTM

Linear Algebra

Springer

Undergraduate Texts in Mathematics

For further volumes:
http://www.springer.com/series/666

Peter Petersen

Linear Algebra

Springer

Peter Petersen
Department of Mathematics
University of California
Los Angeles, CA
USA

ISSN 0172-6056
ISBN 978-1-4614-3611-9 ISBN 978-1-4614-3612-6 (eBook)
DOI 10.1007/978-1-4614-3612-6
Springer New York Heidelberg Dordrecht London

Library of Congress Control Number: 2012938237

Mathematics Subject Classification (2000): 15-01

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

This book covers the aspects of linear algebra that are included in most advanced undergraduate texts. All the usual topics from complex vector spaces, complex inner products the spectral theorem for normal operators, dual spaces, quotient spaces, the minimal polynomial, the Jordan canonical form, and the Frobenius (or rational) canonical form are explained. A chapter on determinants has been included as the last chapter, but they are not used in the text as a whole. A different approach to linear algebra that does not use determinants can be found in [Axler].

The expected prerequisites for this book would be a lower division course in matrix algebra. A good reference for this material is [Bretscher].

In the context of other books on linear algebra it is my feeling that this text is about on a par in difficulty with books such as [Axler, Curtis, Halmos, Hoffman-Kunze, Lang]. If you want to consider more challenging texts, I would suggest looking at the graduate level books [Greub, Roman, Serre].

Chapter 1 contains all of the basic material on abstract vector spaces and linear maps. The dimension formula for linear maps is the theoretical highlight. To facilitate some more concrete developments we cover matrix representations, change of basis, and Gauss elimination. Linear independence which is usually introduced much earlier in linear algebra only comes towards to the end of the chapter. But it is covered in great detail there. We have also included two sections on dual spaces and quotient spaces that can be skipped.

Chapter 2 is concerned with the theory of linear operators. Linear differential equations are used to motivate the introduction of eigenvalues and eigenvectors, but this motivation can be skipped. We then explain how Gauss elimination can be used to compute the eigenvalues as well as the eigenvectors of a matrix. This is used to understand the basics of how and when a linear operator on a finite-dimensional space is diagonalizable. We also introduce the minimal polynomial and use it to give the classic characterization of diagonalizable operators. In the later sections we give a fairly simple proof of the Cayley–Hamilton theorem and the cyclic subspace decomposition. This quickly leads to the Frobenius canonical form. This canonical form is our most general result on how to find a simple matrix representation for a linear map in case it is not diagonalizable. The antepenultimate section explains how the Frobenius canonical form implies the Jordan–Chevalley decomposition and the Jordan-Weierstrass canonical form. In the last section, we present a quick and elementary approach to the Smith normal form. This form allows us to calculate directly all of the similarity invariants of a matrix using basic row and column operations on matrices with polynomial entries.

Chapter 3 includes material on inner product spaces. The Cauchy–Schwarz inequality and its generalization to Bessel's inequality and how they tie in with orthogonal projections form the theoretical centerpiece of this chapter. Along the way, we cover standard facts about orthonormal bases and their existence through the Gram–Schmidt procedure as well as orthogonal complements and orthogonal projections. The chapter also contains the basic elements of adjoints of linear maps and some of its uses to orthogonal projections as this ties in nicely with orthonormal bases. We end the chapter with a treatment of matrix exponentials and systems of differential equations.

Chapter 4 covers quite a bit of ground on the theory of linear maps between inner product spaces. The most important result is of course the spectral theorem for self-adjoint operators. This theorem is used to establish the canonical forms for real and complex normal operators, which then gives the canonical form for unitary, orthogonal, and skew-adjoint operators. It should be pointed out that the proof of the spectral theorem does not depend on whether we use real or complex scalars nor does it rely on the characteristic or minimal polynomials. The reason for ignoring our earlier material on diagonalizability is that it is desirable to have a theory that more easily generalizes to infinite dimensions. The usual proofs that use the characteristic and minimal polynomials are relegated to the exercises. The last sections of the chapter cover the singular value decomposition, the polar decomposition, triangulability of complex linear operators (Schur's theorem), and quadratic forms.

Chapter 5 covers determinants. At this point, it might seem almost useless to introduce the determinant as we have covered the theory without needing it much. While not indispensable, the determinant is rather useful in giving a clean definition for the characteristic polynomial. It is also one of the most important invariants of a finite-dimensional operator. It has several nice properties and gives an excellent criterion for when an operator is invertible. It also comes in handy in giving a formula (Cramer's rule) for solutions to linear systems. Finally, we discuss its uses in the theory of linear differential equations, in particular in connection with the variation of parameters formula for the solution to inhomogeneous equations. We have taken the liberty of defining the determinant of a linear operator through the use of volume forms. Aside from showing that volume forms exist, this gives a rather nice way of proving all the properties of determinants without using permutations. It also has the added benefit of automatically giving the permutation formula for the determinant and hence showing that the sign of a permutation is well defined.

An * after a section heading means that the section is not necessary for the understanding of other sections without an *.

Let me offer a few suggestions for how to teach a course using this book. My assumption is that most courses are based on 150 min of instruction per week with a problem session or two added. I realize that some courses meet three times while others only two, so I will not suggest how much can be covered in a lecture.

First, let us suppose that you, like me, teach in the pedagogically impoverished quarter system: It should be possible to teach Chap. 1, Sects. 1.2–1.13 in 5 weeks, being a bit careful about what exactly is covered in Sects. 1.12 and 1.13. Then, spend 2 weeks on Chap. 2, Sects. 2.3–2.5, possibly omitting Sect. 2.4 covering the minimal polynomial if timing looks tight. Next spend 2 weeks on Chap. 3, Sects. 3.1–3.5, and finish the course by covering Chap. 4, Sect. 4.1 as well as Exercise 9 in Sect. 4.1. This finishes the course with a proof of the Spectral Theorem for self-adjoint operators, although not the proof I would recommend for a more serious treatment.

Next, let us suppose that you teach in a short semester system, as the ones at various private colleges and universities. You could then add 2 weeks of material by either covering the canonical forms from Chap. 2, Sects. 2.6–2.8 or alternately spend 2 weeks covering some of the theory of linear operators on inner product spaces from Chap. 4, Sects. 4.1–4.5. In case you have 15 weeks at your disposal, it might be possible to cover both of these topics rather than choosing between them.

Finally, should you have two quarters, like we sometimes do here at UCLA, then you can in all likelihood cover virtually the entire text. I would certainly recommend that you cover all of Chap. 4 and the canonical form sections in Chap. 2, Sects. 2.6–2.8, as well as the chapter on determinants. If time permits, it might even be possible to include Sects. 2.2, 3.7, 4.8, and 5.8 that cover differential equations.

This book has been used to teach a bridge course on linear algebra at UCLA as well as a regular quarter length course. The bridge course was funded by a VIGRE NSF grant, and its purpose was to ensure that incoming graduate students had really learned all of the linear algebra that we expect them to know when starting graduate school. The author would like to thank several UCLA students for suggesting various improvements to the text: Jeremy Brandman, Sam Chamberlain, Timothy Eller, Clark Grubb, Vanessa Idiarte, Yanina Landa, Bryant Mathews, Shervin Mosadeghi, and Danielle O'Donnol. I am also pleased to acknowledge NSF support from grants DMS 0204177 and 1006677.

I would also like to thank Springer-Verlag for their interest and involvement in this book as well as their suggestions for improvements.

Finally, I am immensely grateful to Joe Borzellino at Cal Poly San Luis Obispo who used the text several times at his institution and supplied me with numerous corrections.

Contents

Chapter 1
Basic Theory

In the first chapter, we are going to cover the definitions of vector spaces, linear maps, and subspaces. In addition, we are introducing several important concepts such as basis, dimension, direct sum, matrix representations of linear maps, and kernel and image for linear maps. We shall prove the dimension theorem for linear maps that relate the dimension of the domain to the dimensions of kernel and image. We give an account of Gauss elimination and how it ties in with the more abstract theory. This will be used to define and compute the characteristic polynomial in Chap. 2.

It is important to note that Sects. 1.13 and 1.12 contain alternate proofs of some of the important results in this chapter. As such, some people might want to go right to these sections after the discussion on isomorphism in Sect. 1.8 and then go back to the missed sections.

As induction is going to play a big role in many of the proofs, we have chosen to say a few things about that topic in the first section.

1.1 Induction and Well-Ordering*

A fundamental property of the *natural numbers*, i.e., the positive integers $\mathbb{N} = \{1, 2, 3, \ldots\}$, that will be used throughout the book is the fact that they are *well ordered*. This means that any nonempty subset $S \subset \mathbb{N}$ has a smallest element $s_{\min} \in S$ such that $s_{\min} \leq s$ for all $s \in S$. Using the natural ordering of the integers, rational numbers, or real numbers, we see that this property does not hold for those numbers. For example, the half-open interval $(0, \infty)$ does not have a smallest element.

In order to justify that the positive integers are well ordered, let $S \subset \mathbb{N}$ be nonempty and select $k \in S$. Starting with 1, we can check whether it belongs to S. If it does, then $s_{\min} = 1$. Otherwise, check whether 2 belongs to S. If $2 \in S$ and

P. Petersen, *Linear Algebra*, Undergraduate Texts in Mathematics,
DOI 10.1007/978-1-4614-3612-6_1,

$1 \notin S$, then we have $s_{\min} = 2$. Otherwise, we proceed to check whether 3 belongs to S. Continuing in this manner, we must eventually find $k_0 \leq k$, such that $k_0 \in S$, but $1, 2, 3, \ldots, k_0 - 1 \notin S$. This is the desired minimum: $s_{\min} = k_0$.

We shall use the well-ordering of the natural numbers in several places in this text. A very interesting application is to the proof of the prime factorization theorem: any integer ≥ 2 is a product of prime numbers. The proof works the following way. Let $S \subset \mathbb{N}$ be the set of numbers which do not admit a prime factorization. If S is empty, we are finished; otherwise, S contains a smallest element $n = s_{\min} \in S$. If n has no divisors, then it is a prime number and hence has a prime factorization. Thus, n must have a divisor $p > 1$. Now write $n = p \cdot q$. Since $p, q < n$ both numbers must have a prime factorization. But then also $n = p \cdot q$ has a prime factorization. This contradicts that S is nonempty.

The second important idea that is tied to the natural numbers is that of *induction*. Sometimes, it is also called *mathematical induction* so as not to confuse it with the inductive method from science. The types of results that one can attempt to prove with induction always have a statement that needs to be verified for each number $n \in \mathbb{N}$. Some good examples are

1. $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.
2. Every integer ≥ 2 has a prime factorization.
3. Every polynomial has a root.

The first statement is pretty straightforward to understand. The second is a bit more complicated, and we also note that in fact, there is only a statement for each integer ≥ 2. This could be finessed by saying that each integer $n + 1$, $n \geq 1$ has a prime factorization. This, however, seems too pedantic and also introduces extra and irrelevant baggage by using addition. The third statement is obviously quite different from the other two. For one thing, it only stands a chance of being true if we also assume that the polynomials have degree ≥ 1. This gives us the idea of how this can be tied to the positive integers. The statement can be paraphrased as: Every polynomial of degree ≥ 1 has a root. Even then, we need to be more precise as $x^2 + 1$ does not have any real roots.

In order to explain how induction works abstractly, suppose that we have a statement $P(n)$ for each $n \in \mathbb{N}$. Each of the above statements can be used as an example of what $P(n)$ can be. The induction process now works by first ensuring that the anchor statement is valid. In other words, we first check that $P(1)$ is true. We then have to establish the *induction step*. This means that we need to show that if $P(n-1)$ is true, then $P(n)$ is also true. The assumption that $P(n-1)$ is true is called the *induction hypothesis*. If we can establish the validity of these two facts, then $P(n)$ must be true for all n. This follows from the well-ordering of the natural numbers. Namely, let $S = \{n : P(n) \text{ is false}\}$. If S is empty, we are finished, otherwise, S has a smallest element $k \in S$. Since $1 \notin S$, we know that $k > 1$. But this means that we know that $P(k-1)$ is true. The induction step then implies that $P(k)$ is true as well. This contradicts that S is nonempty.

Let us see if we can use this procedure on the above statements. For 1, we begin by checking that $1 = \frac{1(1+1)}{2}$. This is indeed true. Next, we assume that

$$1 + 2 + 3 + \cdots + (n-1) = \frac{(n-1)\,n}{2},$$

and we wish to show that

$$1 + 2 + 3 + \cdots + n = \frac{n\,(n+1)}{2}.$$

Using the induction hypothesis, we see that

$$\begin{aligned}(1 + 2 + 3 + \cdots + (n-1)) + n &= \frac{(n-1)\,n}{2} + n \\ &= \frac{(n-1)\,n + 2n}{2} \\ &= \frac{(n+1)\,n}{2}.\end{aligned}$$

Thus, we have shown that $P(n)$ is true provided $P(n-1)$ is true.

For 2, we note that two is a prime number and hence has a prime factorization. Next, we have to prove that n has a prime factorization if $(n-1)$ does. This, however, does not look like a very promising thing to show. In fact, we need a stronger form of induction to get this to work.

The induction step in the stronger version of induction is as follows: If $P(k)$ is true for all $k < n$, then $P(n)$ is also true. Thus, the induction hypothesis is much stronger as we assume that all statements prior to $P(n)$ are true. The proof that this form of induction works is virtually identical to the above justification.

Let us see how this stronger version can be used to establish the induction step for 2. Let $n \in \mathbb{N}$, and assume that all integers below n have a prime factorization. If n has no divisors other than 1 and n, it must be a prime number and we are finished. Otherwise, $n = p \cdot q$ where $p, q < n$. Whence, both p and q have prime factorizations by our induction hypothesis. This shows that also n has a prime factorization.

We already know that there is trouble with statement 3. Nevertheless, it is interesting to see how an induction proof might break down. First, we note that all polynomials of degree 1 look like $ax + b$ and hence have $-\frac{b}{a}$ as a root. This anchors the induction. To show that all polynomials of degree n have a root, we need to first decide which of the two induction hypotheses are needed. There really is not anything wrong by simply assuming that all polynomials of degree $< n$ have a root. In this way, we see that at least any polynomial of degree n that is the product of two polynomials of degree $< n$ must have a root. This leaves us with the so-called prime or irreducible polynomials of degree n, namely, those polynomials that are not divisible by polynomials of degree ≥ 1 and $< n$. Unfortunately, there is not much we can say about these polynomials. So induction does not seem to work well in this case. All is not lost however. A careful inspection of the "proof" of 3 can be modified to show that any polynomial has a prime factorization. This is studied further in Sect. 2.1.

The type of statement and induction argument that we will encounter most often in this text is definitely of the third type. That is to say, it certainly will never be of the very basic type seen in statement 1. Nor will it be as easy as in statement 2. In our cases, it will be necessary to first find the integer that is used for the induction, and even then, there will be a whole collection of statements associated with that integer. This is what is happening in the third statement. There, we first need to select the degree as our induction integer. Next, there are still infinitely many polynomials to consider when the degree is fixed. Finally, whether or not induction will work or is the "best" way of approaching the problem might actually be questionable.

The following statement is fairly typical of what we shall see: Every subspace of $\mathbb{R}^n$ admits a basis with $\leq n$ elements. The induction integer is the dimension n, and for each such integer, there are infinitely many subspaces to be checked. In this case, an induction proof will work, but it is also possible to prove the result without using induction.

1.2 Elementary Linear Algebra

Our first picture of what vectors are and what we can do with them comes from viewing them as geometric objects in the plane and space. Simply put, a vector is an arrow of some given length drawn in the plane. Such an arrow is also known as an oriented line segment. We agree that vectors that have the same length and orientation are equivalent no matter where they are based. Therefore, if we base them at the origin, then vectors are determined by their endpoints. Using a parallelogram, we can add such vectors (see Fig. 1.1). We can also multiply them by scalars. If the scalar is negative, we are changing the orientation. The size of the scalar determines how much we are scaling the vector, i.e., how much we are changing its length (see Fig. 1.2).

This geometric picture can also be taken to higher dimensions. The idea of scaling a vector does not change if it lies in space, nor does the idea of how to add vectors, as two vectors must lie either on a line or more generically in a plane. The problem comes when we wish to investigate these algebraic properties further. As an example, think about the associative law

$$(x + y) + z = x + (y + z) .$$

Clearly, the proof of this identity changes geometrically from the plane to space. In fact, if the three vectors do not lie in a plane and therefore span a parallelepiped, then the sum of these three vectors regardless of the order in which they are added

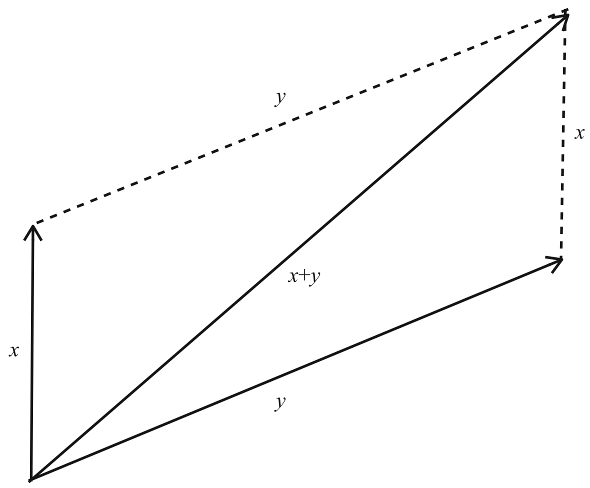

Fig. 1.1 Vector addition

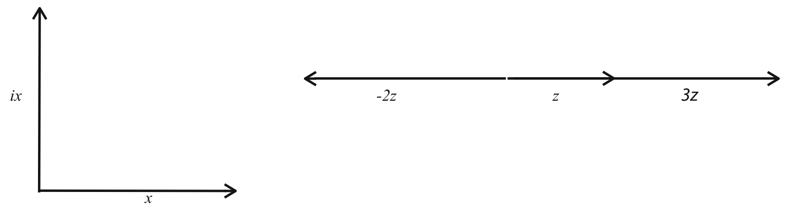

Fig. 1.2 Scalar multiplication

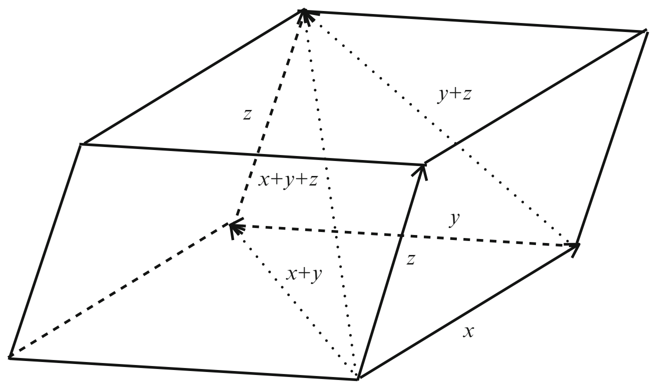

Fig. 1.3 Associativity

is the diagonal of this parallelepiped. The picture of what happens when the vectors lie in a plane is simply a projection of the three-dimensional picture on to the plane (see Fig. 1.3).

The purpose of linear algebra is to clarify these algebraic issues by looking at vectors in a less geometric fashion. This has the added benefit of also allowing other spaces that do not have geometric origins to be included in our discussion. The end result is a somewhat more abstract and less geometric theory, but it has turned out to be truly useful and foundational in almost all areas of mathematics, including geometry, not to mention the physical, natural, and social sciences.

Something quite different and interesting happens when we allow for complex scalars. This is seen in the plane itself which we can interpret as the set of complex numbers. Vectors still have the same geometric meaning, but we can also "scale" them by a number like $i = \sqrt{-1}$. The geometric picture of what happens when multiplying by i is that the vector's length is unchanged as $|i| = 1$, but it is rotated $90°$ (see Fig. 1.2). Thus it is not scaled in the usual sense of the word. However, when we define these notions below, one will not really see any algebraic difference in what is happening. It is worth pointing out that using complex scalars is not just something one does for the fun of it; it has turned out to be quite convenient and important to allow for this extra level of abstraction. This is true not just within mathematics itself. When looking at books on quantum mechanics, it quickly becomes clear that complex vector spaces are the "sine qua non"(without which nothing) of the subject.

1.3 Fields

The "scalars" or numbers used in linear algebra all lie in a *field*. A field is a set $\mathbb{F}$ of numbers, where one has both addition

$$\begin{aligned} \mathbb{F} \times \mathbb{F} &\to \mathbb{F} \\ (\alpha, \beta) &\mapsto \alpha + \beta \end{aligned}$$

and multiplication

$$\begin{aligned} \mathbb{F} \times \mathbb{F} &\to \mathbb{F} \\ (\alpha, \beta) &\mapsto \alpha\beta. \end{aligned}$$

Both operations are assumed associative, commutative, etc. We shall mainly be concerned with the real numbers $\mathbb{R}$ and complex numbers $\mathbb{C}$; some examples will be using the rational numbers $\mathbb{Q}$ as well. These three fields satisfy the axioms we list below.

Definition 1.3.1. A *field* $\mathbb{F}$ is a set whose elements are called numbers or when used in linear algebra *scalars*. The field contains two different elements 0 and 1, and we can add and multiply numbers. These operations satisfy

1. The associative law
$$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma.$$

2. The commutative law

$$\alpha + \beta = \beta + \alpha.$$

3. Addition by 0:

$$\alpha + 0 = \alpha.$$

4. Existence of negative numbers: For each α, we can find $-\alpha$ so that

$$\alpha + (-\alpha) = 0.$$

5. The associative law:

$$\alpha (\beta\gamma) = (\alpha\beta)\gamma.$$

6. The commutative law:

$$\alpha\beta = \beta\alpha.$$

7. Multiplication by 1:

$$\alpha 1 = \alpha.$$

8. Existence of inverses: For each $\alpha \neq 0$, we can find α^{-1} so that

$$\alpha\alpha^{-1} = 1.$$

9. The distributive law:

$$\alpha (\beta + \gamma) = \alpha\beta + \alpha\gamma.$$

One can show that both 0 and 1 are uniquely defined and that the additive inverse $-\alpha$ as well as the multiplicative inverse α^{-1} is unique.

Occasionally, we shall also use that the field has *characteristic zero* this means that

$$n = \overbrace{1 + \cdots + 1}^{n \text{ times}} \neq 0$$

for all positive integers n. Fields such as $\mathbb{F}_2 = \{0, 1\}$ where $1 + 1 = 0$ clearly do not have characteristic zero. We make the assumption throughout the text that all fields have characteristic zero. In fact, there is little loss of generality in assuming that the fields we work are the usual number fields $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$.

There are several important collections of numbers that are not fields:

$$\begin{aligned}
\mathbb{N} &= \{1, 2, 3, \ldots\} \\
&\subset \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\} \\
&\subset \mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\} \\
&= \{0, 1, -1, 2, -2, 3, -3, \ldots\}.
\end{aligned}$$

1.4 Vector Spaces

Definition 1.4.1. A *vector space* consists of a set of vectors V and a field $\mathbb{F}$. The vectors can be added to yield another vector: if $x, y \in V$, then $x + y \in V$ or

$$\begin{aligned} V \times V &\to V \\ (x, y) &\mapsto x + y. \end{aligned}$$

The scalars can be multiplied with the vectors to yield a new vector: if $\alpha \in \mathbb{F}$ and $x \in V$, then $\alpha x \in V$; in other words,

$$\begin{aligned} \mathbb{F} \times V &\to V \\ (\alpha, x) &\mapsto \alpha x. \end{aligned}$$

The vector space contains a *zero vector* 0, also known as the *origin* of V. It is a bit confusing that we use the same symbol for $0 \in V$ and $0 \in \mathbb{F}$. It should always be obvious from the context which zero is used. We shall generally use the notation that scalars, i.e., elements of $\mathbb{F}$, are denoted by small Greek letters such as $\alpha, \beta, \gamma, \ldots$, while vectors are denoted by small roman letters such as $x, y, z, \ldots$. Addition and scalar multiplication must satisfy the following axioms:

1. The associative law:
$$(x + y) + z = x + (y + z).$$
2. The commutative law:
$$x + y = y + x.$$
3. Addition by 0:
$$x + 0 = x.$$
4. Existence of negative vectors: For each x, we can find $-x$ such that
$$x + (-x) = 0.$$
5. The associative law for multiplication by scalars:
$$\alpha (\beta x) = (\alpha\beta) x.$$
6. Multiplication by the unit scalar:
$$1x = x.$$
7. The distributive law when vectors are added:
$$\alpha (x + y) = \alpha x + \alpha y.$$

8. The distributive law when scalars are added:

$$(\alpha + \beta)\, x = \alpha x + \beta x.$$

Remark 1.4.2. We shall also allow scalars to be multiplied on the right of the vector:

$$x\alpha = \alpha x$$

The only slight issue with this definition is that we must ensure that associativity still holds. The key to that is that the field of scalars have the property that multiplication in commutative:

$$\begin{aligned} x\,(\alpha\beta) &= (\alpha\beta)\, x \\ &= (\beta\alpha)\, x \\ &= \beta\,(\alpha x) \\ &= (x\alpha)\,\beta \end{aligned}$$

These axioms lead to several "obvious" facts.

Proposition 1.4.3. *Let V be a vector space over a field $\mathbb{F}$. If $x \in V$ and $\alpha \in \mathbb{F}$, then:*

1. $0x = 0$.
2. $\alpha 0 = 0$.
3. $-1x = -x$.
4. If $\alpha x = 0$, *then either* $\alpha = 0$ *or* $x = 0$.

Proof. By the distributive law,

$$0x + 0x = (0 + 0)\, x = 0x.$$

This together with the associative law gives us

$$\begin{aligned} 0x &= 0x + (0x - 0x) \\ &= (0x + 0x) - 0x \\ &= 0x - 0x \\ &= 0. \end{aligned}$$

The second identity is proved in the same manner.
For the third, consider

$$\begin{aligned} 0 &= 0x \\ &= (1 - 1)\, x \\ &= 1x + (-1)\, x \\ &= x + (-1)\, x, \end{aligned}$$

adding $-x$ on both sides then yields

$$-x = (-1)\,x.$$

Finally, if $\alpha x = 0$ and $\alpha \neq 0$, then we have

$$\begin{aligned} x &= \left(\alpha^{-1}\alpha\right) x \\ &= \alpha^{-1}\left(\alpha x\right) \\ &= \alpha^{-1} 0 \\ &= 0. \end{aligned}$$

□

With these matters behind us, we can relax a bit and start adding, subtracting, and multiplying along the lines we are used to from matrix algebra and vector calculus.

Example 1.4.4. The simplest example of a vector space is the *trivial vector space* $V = \{0\}$ that contains only one point, the origin. The vector space operations and axioms are completely trivial as well in this case.

Here are some important examples of vectors spaces.

Example 1.4.5. The most important basic example is undoubtedly the Cartesian n-fold product of the field $\mathbb{F}$:

$$\begin{aligned} \mathbb{F}^n &= \left\{ \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} : \alpha_1, \ldots, \alpha_n \in \mathbb{F} \right\} \\ &= \left\{ (\alpha_1, \ldots, \alpha_n) : \alpha_1, \ldots, \alpha_n \in \mathbb{F} \right\}. \end{aligned}$$

Note that the $n \times 1$ and the n-tuple ways of writing these vectors are equivalent. When writing vectors in a line of text, the n-tuple version is obviously more convenient. The column matrix version, however, conforms to various other natural choices, as we shall see, and carries some extra meaning for that reason. The ith entry α_i in the vector $x = (\alpha_1, \ldots, \alpha_n)$ is called the ith *coordinate* of x.

Vector addition is defined by adding the entries:

$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} = \begin{bmatrix} \alpha_1 + \beta_1 \\ \vdots \\ \alpha_n + \beta_n \end{bmatrix}$$

and likewise with scalar multiplication

$$\alpha \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \alpha\alpha_1 \\ \vdots \\ \alpha\alpha_n \end{bmatrix},$$

The axioms are verified by using the axioms for the field $\mathbb{F}$.

Example 1.4.6. The *space of functions* whose domain is some fixed set S and whose values all lie in the field $\mathbb{F}$ is denoted by $\operatorname{Func}(S, \mathbb{F}) = \{f : S \to \mathbb{F}\}$. Addition and scalar multiplication is defined by

$$\begin{aligned} (\alpha f)(x) &= \alpha f(x), \\ (f_1 + f_2)(x) &= f_1(x) + f_2(x). \end{aligned}$$

And the axioms again follow from using the field axioms for $\mathbb{F}$.

In the special case where $S = \{1, \ldots, n\}$, it is worthwhile noting that

$$\operatorname{Func}(\{1, \ldots, n\}, \mathbb{F}) = \mathbb{F}^n.$$

Thus, vectors in $\mathbb{F}^n$ can also be thought of as functions and can be graphed as either an arrow in space or as a histogram type function. The former is of course more geometric, but the latter certainly also has its advantages as collections of numbers in the form of $n \times 1$ matrices do not always look like vectors. In statistics, the histogram picture is obviously far more useful. The point here is that the way in which vectors are pictured might be psychologically important, but from an abstract mathematical perspective, there is no difference.

Example 1.4.7. The space of $n \times m$ matrices

$$\begin{aligned} \operatorname{Mat}_{n \times m}(\mathbb{F}) &= \left\{ \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1m} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nm} \end{bmatrix} : \alpha_{ij} \in \mathbb{F} \right\} \\ &= \left\{ \left(\alpha_{ij} \right) : \alpha_{ij} \in \mathbb{F} \right\}. \end{aligned}$$

$n \times m$ matrices are evidently just a different way of arranging vectors in $\mathbb{F}^{n \cdot m}$. This arrangement, as with the column version of vectors in $\mathbb{F}^n$, imbues these vectors with some extra meaning that will become evident as we proceed.

Example 1.4.8. There is a slightly more abstract vector space that we can construct out of a general set S and a vector space V. This is the set $\operatorname{Map}(S, V)$ of all maps from S to V. Scalar multiplication and addition are defined as follows:

$$\begin{aligned} (\alpha f)(x) &= \alpha f(x), \\ (f_1 + f_2)(x) &= f_1(x) + f_2(x). \end{aligned}$$

The axioms now follow from V being a vector space.

The space of maps is in some sense the most general type of vector space as all other vector spaces are either of this type or *subspaces* of such function spaces.

Definition 1.4.9. A nonempty subset $M \subset V$ of a vector space V is said to be a *subspace* if it is closed under addition and scalar multiplication:

$$x, y \in M \Rightarrow x + y \in M,$$
$$\alpha \in \mathbb{F} \text{ and } x \in M \Rightarrow \alpha x \in M.$$

We also say that M is *closed* under vector addition and multiplication by scalars.

Note that since $M \neq \emptyset$, we can find $x \in M$; this means that $0 = 0 \cdot x \in M$. Thus, subspaces become vector spaces in their own right and this without any further checking of the axioms.

Example 1.4.10. The set of *polynomials* whose coefficients lie in the field $\mathbb{F}$

$$\mathbb{F}[t] = \left\{p(t) = a_0 + a_1 t + \cdots + a_k t^k : k \in \mathbb{N}_0, a_0, a_1, \ldots, a_k \in \mathbb{F}\right\}$$

is also a vector space. If we think of polynomials as functions, then we imagine them as a subspace of $\mathrm{Func}(\mathbb{F}, \mathbb{F})$. However, the fact that a polynomial is determined by its representation as a function depends on the fact that we have a field of characteristic zero! If, for instance, $\mathbb{F} = \{0, 1\}$, then the polynomial $t^2 + t$ vanishes when evaluated at both 0 and 1. Thus, this nontrivial polynomial is, when viewed as a function, the same as $p(t) = 0$.

We could also just record the coefficients. In that case, $\mathbb{F}[t]$ is a subspace of $\mathrm{Func}(\mathbb{N}_0, \mathbb{F})$ and consists of those infinite tuples that are zero except at all but a finite number of places.

If

$$p(t) = a_0 + a_1 t + \cdots + a_n t^n \in \mathbb{F}[t],$$

then the largest integer $k \leq n$ such that $a_k \neq 0$ is called the degree of p. In other words,

$$p(t) = a_0 + a_1 t + \cdots + a_k t^k$$

and $a_k \neq 0$. We use the notation $\deg(p) = k$.

Example 1.4.11. The collection of formal power series

$$\begin{aligned} \mathbb{F}[[t]] &= \left\{a_0 + a_1 t + \cdots + a_k t^k + \cdots : a_0, a_1, \ldots, a_k, \ldots \in \mathbb{F}\right\} \\ &= \left\{\sum_{i=0}^{\infty} a_i t^i : a_i \in \mathbb{F}, i \in \mathbb{N}_0\right\} \end{aligned}$$

bears some resemblance to polynomials, but without further discussions on convergence or even whether this makes sense, we cannot interpret power series as lying in $\mathrm{Func}(\mathbb{F}, \mathbb{F})$. If, however, we only think about recording the coefficients, then we see that $\mathbb{F}[[t]] = \mathrm{Func}(\mathbb{N}_0, \mathbb{F})$. The extra piece of information that both $\mathbb{F}[t]$ and $\mathbb{F}[[t]]$ carry with them, aside from being vector spaces, is that the elements can also be multiplied. This extra structure will be used in the case of $\mathbb{F}[t]$. Power series will not play an important role in the sequel. Finally, note that $\mathbb{F}[t]$, is a subspace of $\mathbb{F}[[t]]$.

Example 1.4.12. For two (or more) vector spaces V, W over the same field $\mathbb{F}$ we can form the (Cartesian) product

$$V \times W = \{(v, w) : v \in V \text{ and } w \in W\}.$$

Scalar multiplication and addition are defined by

$$\alpha (v, w) = (\alpha v, \alpha w),$$
$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2).$$

Note that $V \times W$ is not in a natural way a subspace in a space of functions or maps.

Exercises

1. Find a subset $C \subset \mathbb{F}^2$ that is closed under scalar multiplication but not under addition of vectors.
2. Find a subset $A \subset \mathbb{C}^2$ that is closed under vector addition but not under multiplication by complex numbers.
3. Find a subset $Q \subset \mathbb{R}$ that is closed under addition but not scalar multiplication by real scalars.
4. Let $V = \mathbb{Z}$ be the set of integers with the usual addition as "vector addition." Show that it is not possible to define scalar multiplication by $\mathbb{Q}, \mathbb{R}$, or $\mathbb{C}$ so as to make it into a vector space.
5. Let V be a real vector space, i.e., a vector space were the scalars are $\mathbb{R}$. The *complexification* of V is defined as $V_{\mathbb{C}} = V \times V$. As in the construction of complex numbers, we agree to write $(v, w) \in V_{\mathbb{C}}$ as $v + iw$. Moreover, if $v \in V$, then it is convenient to use the shorthand notations $v = v + i0$ and $iv = 0 + iv$. Define complex scalar multiplication on $V_{\mathbb{C}}$ and show that it becomes a complex vector space.
6. Let V be a complex vector space i.e., a vector space were the scalars are $\mathbb{C}$. Define V^* as the complex vector space whose additive structure is that of V but where complex scalar multiplication is given by $\lambda * x = \bar{\lambda} x$. Show that V^* is a complex vector space.
7. Let P_n be the set of polynomials in $\mathbb{F}[t]$ of degree $\leq n$.
   (a) Show that P_n is a vector space.
   (b) Show that the space of polynomials of degree $n \geq 1$ is $P_n - P_{n-1}$ and does not form a subspace.
   (c) If $f(t) : \mathbb{F} \to \mathbb{F}$, show that $V = \{p(t) f(t) : p \in P_n\}$ is a subspace of $\mathrm{Func}(\mathbb{F}, \mathbb{F})$.

sample content of Linear Algebra and Differential Equations

- download Utvecklingspsykologi
- Sample Size Calculations in Clinical Research (2nd Edition) (Chapman & Hall/CRC Biostatistics Series) pdf, azw (kindle), epub
- read A Little Lumpen Novelita
- read online Crimes of the Educators: How Utopians Are Using Government Schools to Destroy America's Children

- http://growingsomeroots.com/ebooks/Utvecklingspsykologi.pdf
- http://fortune-touko.com/library/Midnight-Thief--Midnight-Thief--Book-1-.pdf
- http://academialanguagebar.com/?ebooks/Telling-the-Truth-about-History.pdf
- http://betsy.wesleychapelcomputerrepair.com/library/The-Dinnertime-Survival-Cookbook--Delicious--Inspiring-Meals-for-Busy-Families.pdf