CHAPMAN & HALL/CRC
Monographs and Surveys in
Pure and Applied Mathematics 144

HIGHER ORDER DERIVATIVES

Satya N. Mukhopadhyay
in collaboration with P. S. Bullen

CRC Press
Taylor & Francis Group
A CHAPMAN & HALL BOOK

CHAPMAN & HALL/CRC
Monographs and Surveys in
Pure and Applied Mathematics 144

HIGHER ORDER DERIVATIVES

CHAPMAN & HALL/CRC
Monographs and Surveys in Pure and Applied Mathematics

Main Editors
H. Brezis, *Université de Paris*
R.G. Douglas, *Texas A&M University*
A. Jeffrey, *University of Newcastle upon Tyne (Founding Editor)*

Editorial Board
R. Aris, *University of Minnesota*
G.I. Barenblatt, *University of California at Berkeley*
H. Begehr, *Freie Universität Berlin*
P. Bullen, *University of British Columbia*
R.J. Elliott, *University of Alberta*
R.P. Gilbert, *University of Delaware*
R. Glowinski, *University of Houston*
D. Jerison, *Massachusetts Institute of Technology*
K. Kirchgässner, *Universität Stuttgart*
B. Lawson, *State University of New York*
B. Moodie, *University of Alberta*
L.E. Payne, *Cornell University*
D.B. Pearson, *University of Hull*
G.F. Roach, *University of Strathclyde*
I. Stakgold, *University of Delaware*
W.A. Strauss, *Brown University*
J. van der Hoek, *University of Adelaide*

CHAPMAN & HALL/CRC
Monographs and Surveys in
Pure and Applied Mathematics 144

HIGHER ORDER DERIVATIVES

Satya N. Mukhopadhyay
in collaboration with P. S. Bullen

CRC Press
Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group, an **informa** business
A CHAPMAN & HALL BOOK

CRC Press
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

© 2012 by Taylor & Francis Group, LLC
CRC Press is an imprint of Taylor & Francis Group, an Informa business

No claim to original U.S. Government works
Version Date: 20111215

International Standard Book Number-13: 978-1-4398-8048-7 (eBook - PDF)

This book contains information obtained from authentic and highly regarded sources. Reasonable efforts have been made to publish reliable data and information, but the author and publisher cannot assume responsibility for the validity of all materials or the consequences of their use. The authors and publishers have attempted to trace the copyright holders of all material reproduced in this publication and apologize to copyright holders if permission to publish in this form has not been obtained. If any copyright material has not been acknowledged please write and let us know so we may rectify in any future reprint.

Except as permitted under U.S. Copyright Law, no part of this book may be reprinted, reproduced, transmitted, or utilized in any form by any electronic, mechanical, or other means, now known or hereafter invented, including photocopying, microfilming, and recording, or in any information storage or retrieval system, without written permission from the publishers.

For permission to photocopy or use material electronically from this work, please access www.copyright.com (http://www.copyright.com/) or contact the Copyright Clearance Center, Inc. (CCC), 222 Rosewood Drive, Danvers, MA 01923, 978-750-8400. CCC is a not-for-profit organization that provides licenses and registration for a variety of users. For organizations that have been granted a photocopy license by the CCC, a separate system of payment has been arranged.

Trademark Notice: Product or corporate names may be trademarks or registered trademarks, and are used only for identification and explanation without intent to infringe.

Visit the Taylor & Francis Web site at
http://www.taylorandfrancis.com

and the CRC Press Web site at
http://www.crcpress.com

This book is dedicated to

My Family

This page intentionally left blank

Contents

This page intentionally left blank

Preface

The concept of the first-order ordinary derivative is as old as the invention of the calculus. Since then the concept of the first-order ordinary derivative has been extended to higher order derivatives in various directions. The nth-order ordinary derivative $f^{(n)}$ of a function f is the first order derivative of its $(n-1)$st-order derivative $f^{(n-1)}$ and has no special importance. But, the higher order derivatives other than the ordinary one are particularly interesting in general because they are derivatives for which the nth order derivative can exist without the $(n-1)$st-order derivative existing. For instance, the classical Riemann derivative is an example of this type and plays an important role in the theory of trigonometric series.

Higher order derivatives of different types have been considered by several authors with the results appearing in various journals over a long period of time. We consider these higher order derivatives and study the relations between them. It is hoped that the resulting monograph will be particularly helpful to those young mathematicians who wish to pursue their studies in this branch of real analysis.

We cannot claim to have considered all known types of higher order derivatives and suggestions for further inclusion and improvements in general are welcome.

This page intentionally left blank

Introduction

The concept of higher order derivatives is useful in many branches of mathematics and its applications. Among the higher order derivatives, the Peano derivative is the most well known. This notion started from the viewpoint of approximating a function by polynomials and its origin is in the paper of G. Peano [139]. This derivative was subsequently considered, under various different names, by A. Genocchi, [68] and C.J. de la Vallée Poussin [174], and later by J.C. Burkill [28], A. Denjoy [45], J. Marcinkiewicz & A. Zygmund [106], J. Marcinkiewicz [105], A. Zygmund [194], and E. Corominas [37]. Finally in 1954, H.W. Oliver [134] gave the concept the name "Peano derivative" and made a systematic study of the Peano derivative. After this C.E. Weil [184] began to work on this derivative and since then extensive work has been done by Weil and many other authors [3, 10, 13–15, 17, 19, 23, 24, 26, 27, 33, 41, 52, 57, 59, 60, 63, 65, 70, 71, 73, 74, 82, 93–95, 97, 100–102, 124, 127, 128, 135, 137, 138, 142, 151, 159, 161, 181, 183, 185, 195, 196].

The Riemann* derivative is considered by A. Denjoy [45] and E. Corominas [37] under the name generalized derivative. They proved that this derivative exists finitely if and only if the Peano derivative exists finitely, and then the values are equal. The infinite case is considered in [26]; in this paper, there are some lacunæ pointed out in [94], which are filled up in [197]. A.M. Russel [145] gave this derivative its present name, Riemann* derivative, and used it to study functions of bounded variations of higher order. This derivative has been found to be useful as well for the study of functions of higher-order absolute continuity [43, 114].

The de la Vallée Poussin derivative, also called the symmetric Peano derivative, was introduced under the name generalized derivative by C.J. de la Vallée Poussin himself [174]; he used it to study various properties of Fourier series. Since then, this derivative has been studied and used by many authors [7, 9, 16, 20, 22, 25, 30, 34, 36, 38, 47, 72, 77, 78, 84, 87–92, 105, 108–110, 115, 116, 119, 121–123, 128, 144, 146, 147, 155, 159, 162, 165–168, 172, 173, 193, 194, 196].

The Cesàro derivative originated from the Cesàro summability of series. Introduced by J.C. Burkill to define the Cesàro–Perron integral [28, 29], these derivatives were studied in detail by W.L.C. Sargent [150–152]. In addition, there is the work of J.A. Bergin [10], who studied the Cesàro derivatives proving, in certain cases, their equivalence to the Peano derivatives (see also [99]).

J.C. Burkill later introduced symmetric Cesàro derivatives [30] to define the symmetric Cesàro–Perron integral, which is useful in solving the so-called coefficient problem in the theory of trigonometric series; see also [25]. Higher order symmetric Cesàro derivatives were introduced in [22] to define the SC_nP-integral (see also [38]).

The concept of the Borel derivative was introduced by E. Borel, calling it the average derivative (derivée moyenne.) First-order Borel derivatives, both unsymmetric and symmetric, were studied by A. Khintchine [86] and by J. Marcinkiewicz & A. Zygmund [106]. Then W.L.C. Sargent [149] made an intensive study of the first-order unsymmetric Borel derivative (see also C.J. Neugebauer [133]). The symmetric Borel derivative was used by J. Marcinkiewicz & A. Zygmund [106] and by P.S. Bullen & S.N. Mukhopadhyay [25] to consider a trigonometric integral.

The L_p-derivative originated from the work of A.P. Calderón & A. Zygmund [32] and was considered in [159]. This derivative appears in [2, 5, 50, 58, 133,136,189,196]. The symmetric L_p-derivative is considered in [2,52,131,190].

The Abel derivative has a very special nature. Its origin can be found in a dormant state in the theory of trigonometric series and occurred in the work of A. Rajchman and A. Zygmund when they considered the Riemann and Abel summability of trigonometric series (see [196; p. 353, Lemma 7.6]). The definition of the Abel derivative is not given there, but is defined by S.J. Taylor [163]. Taylor only defined the second-order derivative and used it to introduce his Abel–Perron integral, which is helpful in solving the coefficient problem for Abel summable trigonometric series. As in the case of the de la Vallée Poussin derivative, the existence of the second-order Abel derivative of a function at a point does not imply the existence of the first-order Abel derivative at that point. To help with this, the concept of Abel smoothness is defined. The seeds of this and of higher-order Abel derivatives are in [122] and [121], respectively, and they are fully defined in the text.

The Laplace derivative is introduced in [160, 161]; its theory is still being developed [198]. Symmetric Laplace derivatives are introduced in the text in a very natural way.

The symmetric Riemann derivative was found to be convenient for studying the Riemann summability of trigonometric series. Both unsymmetric and symmetric Riemann derivatives of higher order are considered and studied by J. Marcinkiewicz [105] and J. Marcinkiewicz & A. Zygmund [106]. Symmetric Riemann derivatives of higher order are the special study of P.L. Butzer & W. Kozakiewicz [31] and T.K. Dutta & S.N. Mukhopadhyay [48]. Further work on both unsymmetric and symmetric Riemann derivatives of higher order can be found in [71,73,74,81–83,125,148,175–180,182,196]. J. Marcinkiewicz & A. Zygmund [106] defined another derivative of higher order that is of interest, which we call the MZ-derivative.

J.M. Ash [2], following a suggestion of A. Zygmund, introduces a system, a finite set of real numbers satisfying certain conditions, and uses this system to define a generalized derivative of higher order. Suitable specializations of the system lead to the symmetric and unsymmetric Riemann derivatives and to the MZ-derivative. This work is continued in [4,6,64,75,76,126]. The derivative of Ash is such that, if any of the derivatives of Peano or de la Vallée Poussin exists finitely, then the appropriate derivative of Ash also exists with the same value. This equivalence does not extend to the case of derivates, however, although a suitable modification of the Ash definition can be made to allow for this. That is, the modified Ash definition not only includes the Peano and de la Vallée Poussin derivatives, but their corresponding derivates as well.

As has been mentioned, higher order derivatives occur in the study of convexity and bounded variation of higher order (see [1, 16, 21, 35, 39, 40, 69, 70, 85, 111–113, 119, 129, 140, 141, 143, 154, 188, 192] and [11, 43, 114, 130, 145, 166], respectively).

In the case of symmetric derivatives, smoothness is sometimes helpful in various discussions (see above paragraph on the Abel derivative). Thus, if a function f has a de la Vallée Poussin derivative of order n at x, then while f has a derivative of order $n-2$ at x it may not have a derivative of order $n-1$ at x. Then smoothness of order n may compensate for the nonexistence of this derivative of order $n-1$, or if f is not smooth, then it may be quasi-smooth; see [1, 18, 46, 49, 51, 55, 56, 80, 104, 132, 153, 156, 158, 169–171].

We have not considered the approximate analogues of the derivatives and smoothness discussed above, but papers on these topics are included in the bibliography. This bibliography is surely not complete and any suggestion for further inclusion is welcome.

Although much work has been done on the Peano and de la Vallée Poussin derivatives, there is a large amount of work to be done on the other higher order derivatives as their properties remain often virtually unexplored. The purpose of this book is to introduce to any newcomer interested in the field of higher order derivatives to the present state of knowledge. The background required is that of only basic advanced real analysis and, although the special Denjoy integral has been used, a knowledge of the Lebesgue integral should suffice.

The book contains two chapters. Chapter 1 contains 16 subsections where the various higher order derivatives are introduced. Chapter 2 contains 24 subsections where the relations between these derivatives are given.

This page intentionally left blank

Chapter 1

Higher Order Derivatives

1.1 Divided Differences of Order n

Let n be a fixed positive integer and let f be a real valued function defined on the set $\{x_0, \ldots, x_n\}$ of $n+1$ distinct points. The nth order difference is defined by

$$Q_n(f; x_0, \ldots, x_n) = \frac{Q_{n-1}(f; x_0, \ldots, x_{n-1}) - Q_{n-1}(f; x_1, \ldots, x_n)}{x_0 - x_n}, \ n \geq 2, \tag{1.1.1}$$

with

$$Q_1(f; x_i, x_j) = \frac{f(x_i) - f(x_j)}{x_i - x_j}, \ i \neq j, \ i, j = 0, \ldots, n. \tag{1.1.2}$$

A simple inductive argument, given below, shows that

$$Q_n(f; x_0, \ldots, x_n) = \sum_{i=0}^{n} \frac{f(x_i)}{\prod_{\substack{j=0 \\ i \neq j}}^{n} (x_i - x_j)} = \sum_{i=0}^{n} \frac{f(x_i)}{\omega'(x_i)}, \tag{1.1.3}$$

where

$$\omega(x) = \prod_{j=0}^{n} (x - x_j).$$

□ In fact, if $n = 1$, then (1.1.3) follows from (1.1.2). Suppose that (1.1.3) holds for $n = m$ and let $n = m + 1$. Then, from (1.1.1), we have, since (1.1.3) holds for $n = m$,

$$\begin{aligned}
&(x_0 - x_{m+1}) Q_{m+1}(f; x_0, \ldots, x_{m+1}) \\
&\quad = Q_m(f; x_0, \ldots, x_m) - Q_m(f; x_1, \ldots, x_{m+1}) \\
&\quad = \sum_{i=0}^{m} \frac{f(x_i)}{\prod_{\substack{j=0 \\ i \neq j}}^{m} (x_i - x_j)} - \sum_{i=1}^{m+1} \frac{f(x_i)}{\prod_{\substack{j=1 \\ i \neq j}}^{m+1} (x_i - x_j)} \\
&\quad = \frac{f(x_0)}{\prod_{j=1}^{m} (x_0 - x_j)} + \sum_{i=1}^{m} \frac{f(x_i)}{\prod_{\substack{j=0 \\ i \neq j}}^{m} (x_i - x_j)} \\
&\qquad - \sum_{i=1}^{m} \frac{f(x_i)}{\prod_{\substack{j=1 \\ i \neq j}}^{m+1} (x_i - x_j)} - \frac{f(x_{m+1})}{\prod_{j=1}^{m} (x_{m+1} - x_j)}
\end{aligned}$$

$$
\begin{aligned}
&= (x_0 - x_{m+1})\frac{f(x_0)}{\prod_{j=1}^{m+1}(x_0 - x_j)} + \sum_{i=1}^{m}\frac{f(x_i)(x_i - x_{m+1} - x_i + x_0)}{\prod_{\substack{j=0\\ i\neq j}}^{m+1}(x_i - x_j)} \\
&\quad - \frac{f(x_{m+1})(x_{m+1} - x_0)}{\prod_{j=0}^{m}(x_{m+1} - x_j)} \\
&= (x_0 - x_{m+1})\left[\frac{f(x_0)}{\prod_{j=1}^{m+1}(x_0 - x_j)} + \sum_{i=1}^{m}\frac{f(x_i)}{\prod_{\substack{j=0\\ i\neq j}}^{m+1}(x_i - x_j)} + \frac{f(x_{m+1})}{\prod_{j=0}^{m}(x_{m+1} - x_j)}\right] \\
&= (x_0 - x_{m+1})\sum_{i=0}^{m+1}\frac{f(x_i)}{\prod_{\substack{j=0\\ i\neq j}}^{m+1}(x_i - x_j)}
\end{aligned}
$$

and so (1.1.3) is established for $n = m + 1$. □

It follows, from (1.1.3), that for any two functions f and g and for any two real numbers α and β

$$
Q_n(\alpha f + \beta g; x_0, \ldots, x_n) = \alpha Q_n(f; x_0, \ldots, x_n) + \beta Q_n(g; x_0, \ldots, x_n). \quad (1.1.4)
$$

It follows, also from (1.1.3), that $Q_n(f; x_0, \ldots, x_n)$ is independent of the order of the points $x_0, \ldots, x_n$.

We show that

$$
Q_n(f; x_0, \ldots, x_n) = \frac{\begin{vmatrix} 1 & 1 & \ldots & 1 \\ x_0 & x_1 & \ldots & x_n \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ x_0^{n-1} & x_1^{n-1} & \ldots & x_n^{n-1} \\ f(x_0) & f(x_1) & \ldots & f(x_n) \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \ldots & 1 \\ x_0 & x_1 & \ldots & x_n \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ x_0^{n-1} & x_1^{n-1} & \ldots & x_n^{n-1} \\ x_0^{n} & x_1^{n} & \ldots & x_n^{n} \end{vmatrix}} = \frac{D\big(f(x)\big)}{D(x^n)}, \text{ say.} \quad (1.1.5)
$$

□ If D_r is the co-factor of $f(x_r)$ in $D\big(f(x)\big)$, then

$$
D_r = (-1)^{n+r+2}\begin{vmatrix} 1 & 1 & \ldots & 1 & 1 & \ldots & 1 \\ x_0 & x_1 & \ldots & x_{r-1} & x_{r+1} & \ldots & x_n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_0^{n-1} & x_1^{n-1} & \ldots & x_{r-1}^{n-1} & x_{r+1}^{n-1} & \ldots & x_n^{n-1} \end{vmatrix}. \quad (1.1.6)
$$

Since the determinants D_r and $D(x^n)$ are van der Monde determinants, we have

$$D(x^n) = \prod_{i<j}(x_j - x_i), \tag{1.1.7}$$

$$D_r = (-1)^{n+r+2}{\prod_{i<j}}'(x_j - x_i), \tag{1.1.8}$$

where in (1.1.8) x_r never occurs. Hence, writing all the factors in these two expressions and cancelling equal factors we get

$$\begin{aligned}\frac{D(x^n)}{D_r} &= (-1)^{n+r+2}(x_n - x_r)(x_{n-1} - x_r)\dots \\ &\quad \times (x_{r+1} - x_r)(x_r - x_0)(x_r - x_1)\dots(x_r - x_{r-1}).\end{aligned} \tag{1.1.9}$$

Since D_r is the co-factor of $f(x_r)$ in $D\big(f(x)\big)$, we have from the above that

$$\frac{D\big(f(x)\big)}{D(x^n)} = \sum_{r=0}^{n} \frac{f(x_r)D_r}{D(x^n)} = \sum_{r=0}^{n} \frac{f(x_r)}{\prod_{\substack{j=0\\ r\neq j}}^{n}(x_r - x_j)} = Q_n(f; x_0, \dots, x_n), \tag{1.1.10}$$

which completes the proof of (1.1.5). □

From (1.1.5) it follows that

$$Q_n(x^p; x_0, \dots, x_n) = \begin{cases} 0, & \text{if } p = 0, 1, \dots, n-1, \\ 1, & \text{if } p = n. \end{cases} \tag{1.1.11}$$

Theorem 1.1.1 *If $f\colon \mathbb{R} \mapsto \mathbb{R}$ is a polynomial of degree at most n, then for every choice of distinct points $x_0, \dots, x_n, Q_n(f; x_0, \dots, x_n) = 0$ or a_n according, as the degree of f is less than n or equal to n; a_n being the coefficient of x^n in f.*

□ The proof follows from (1.1.4) and (1.1.11). □

Theorem 1.1.2 *If $f\colon \mathbb{R} \mapsto \mathbb{R}$, then for every choice of distinct points $x_0, \dots, x_n$,*

$$\begin{aligned}&Q_n(f; x_0, \dots, x_n) = \\ &\quad \frac{y - x_0}{x_n - x_0} Q_n(f; x_0, \dots, x_{n-1}, y) + \frac{x_n - y}{x_n - x_0} Q_n(f; y, \dots, x_n),\end{aligned} \tag{1.1.12}$$

where y is a point distinct from $x_0, \dots, x_n$.

□ From (1.1.1),

$$\begin{aligned}&(x_0 - y)Q_n(f; x_0, \dots, x_{n-1}, y) = \\ &\quad Q_{n-1}(f; x_0, \dots, x_{n-1}) - Q_{n-1}(f; x_1, \dots, x_{n-1}, y)\end{aligned} \tag{1.1.13}$$

$$\begin{aligned}&(y - x_n)Q_n(f; y, x_1, \dots, x_n) = \\ &\quad Q_{n-1}(f; y, x_1, \dots, x_{n-1}) - Q_{n-1}(f; x_1, \dots, x_n).\end{aligned} \tag{1.1.14}$$

Adding (1.1.13) and (1.1.14) and using (1.1.1), we get (1.1.12). □

Theorem 1.1.3 *Let* $f\colon \mathbb{R} \mapsto \mathbb{R}$ *and* $x_0, \ldots, x_n$ *be distinct points. If* $y_0, \ldots, y_{r-1}$ *is another set of distinct points, each of which is different from* $x_0, \ldots, x_n$*, rename* $x_0, \ldots, x_n, y_0, \ldots, y_{r-1}$ *as* $z_0, \ldots, z_{n+r}$*. Then there are real numbers* $\alpha_1, \ldots, \alpha_r$ *independent of* f *such that*

$$Q_n(f; x_0, \ldots, x_n) = \sum_{i=0}^{r} \alpha_i Q_n(f; z_i, \ldots, z_{i+n}) \tag{1.1.15}$$

and

$$\sum_{i=0}^{r} \alpha_i = 1. \tag{1.1.16}$$

□ If $r = 1$, the proof follows from Theorem 1.1.2 with $\alpha_0 = (y_0 - x_0)/(x_n - x_0)$ and $\alpha_1 = (x_n - y_0)/(x_n - x_0)$.

We suppose that the theorem is true for $r = r_0$ and consider $r_0 + 1$ distinct points $y_0, \ldots, y_{r_0}$ different from $x_0, \ldots, x_n$. Let the points $x_0, \ldots, x_n, y_0, \ldots, y_{r_0-1}$ be renamed as $z_0, \ldots, z_{n+r_0}$. Since the theorem is true for $r = r_0$, there are numbers $\alpha_0, \ldots, \alpha_{r_0}$ such that

$$Q_n(f; x_0, \ldots, x_n) = \sum_{i=0}^{r_0} \alpha_i Q_n(f; z_i, \ldots, z_{i+n}) \tag{1.1.17}$$

and

$$\sum_{i=0}^{r_0} \alpha_i = 1. \tag{1.1.18}$$

Now applying (1.1.12) replacing $x_0, \ldots, x_n, y$ by $z_{r_0}, \ldots, z_{r_0+n}, y_{r_0}$, respectively, we have

$$\begin{aligned} Q_n(f; z_{r_0}, \ldots, z_{r_0+n}) &= \frac{y_{r_0} - z_{r_0}}{z_{r_0+n} - z_{r_0}} Q_n(f; z_{r_0}, \ldots, z_{r_0+n-1}, y_{r_0}) \\ &\quad + \frac{z_{r_0+n} - y_{r_0}}{z_{r_0+n} - z_{r_0}} Q_n(f; y_{r_0}, z_{r_0}, \ldots, z_{r_0+n}), \end{aligned} \tag{1.1.19}$$

and so from (1.1.17) and (1.1.19)

$$\begin{aligned} Q_n(f; x_0, \ldots, x_n) &= \sum_{i=0}^{r_0-1} \alpha_i Q_n(f; z_i, \ldots, z_{i+n}) + \alpha_{r_0} Q_n(f; z_{r_0}, \ldots, z_{r_0+n}) \\ &= \sum_{i=0}^{r_0-1} \alpha_i Q_n(f; z_i, \ldots, z_{i+n}) + \alpha_{r_0} \frac{y_{r_0} - z_{r_0}}{z_{r_0+n} - z_{r_0}} Q_n(f; z_{r_0}, \ldots, z_{r_0+n-1}, y_{r_0}) \\ &\quad + \alpha_{r_0} \frac{z_{r_0+n} - y_{r_0}}{z_{r_0+n} - z_{r_0}} Q_n(f; y_{r_0}, z_{r_0+1}, \ldots, z_{r_0+n}). \end{aligned} \tag{1.1.20}$$

Now putting $\beta_i = \alpha_i$, $0 \le i \le r_0 - 1, \beta_{r_0} = \alpha_{r_0} \frac{y_{r_0} - z_{r_0}}{z_{r_0+n} - z_{r_0}}, \beta_{r_0+1} = \alpha_{r_0} \frac{z_{r_0+n} - y_{r_0}}{z_{r_0+n} - z_{r_0}}$ we have from (1.1.18)

$$\sum_{i=0}^{r_0+1} \beta_i = \sum_{i=0}^{r_0-1} \alpha_i + \alpha_{r_0} \left[\frac{y_{r_0} - z_{r_0}}{z_{r_0+n} - z_{r_0}} + \frac{z_{r_0+n} - y_{r_0}}{z_{r_0+n} - z_{r_0}} \right] = \sum_{i=0}^{r_0} \alpha_i = 1. \quad (1.1.21)$$

So, renaming the points $z_0, \ldots, z_{r_0+n-1}, y_0, z_{r_0+n}$ as $\omega_0, \ldots, \omega_{n+r_0+1}$ we have from (1.1.20)

$$Q_n(f; x_0 \ldots, x_n) = \sum_{i=0}^{r_0+1} \beta_i Q_n(f; \omega_i, \ldots, \omega_{i+n}). \quad (1.1.22)$$

The relations (1.1.22) and (1.1.21) prove the theorem for $r = r_0 + 1$ and so the theorem is proved by induction. □

Remark. If the points $y_0, \ldots, y_{r-1}$ in Theorem 1.1.3 satisfy

$$\min_{0 \le i \le n} x_i \le y_j \le \max_{0 \le i \le n} x_i,\ j = 0, \ldots, r-1,$$

then all the α_js are positive.

□ This is true for Theorem 1.1.2 and so the remark follows by the induction above; [145]. □

1.2 General Derivatives of Order n

Let n be a fixed positive integer and let $B = \{a_0, \ldots, a_n\}$ where $a_0, \ldots, a_n$ are distinct real numbers. Let $f \colon \mathbb{R} \mapsto \mathbb{R}$, the *general derivative of f at $x \in \mathbb{R}$ with respect to B* is defined by

$$GD_n f(x, B) = \lim_{h \to 0} n! Q_n(f; x + ha_0, \ldots, x + ha_n), \quad (1.2.1)$$

provided the limit on the right-hand side of (1.2.1) exists. So by (1.1.3)

$$\begin{aligned} GD_n f(x, B) &= \lim_{h \to 0} \frac{n!}{h^n} \sum_{i=0}^{n} \frac{f(x + a_i h)}{\prod_{\substack{j=0 \\ j \ne i}}^{n} (a_i - a_j)} \\ &= \lim_{h \to 0} \frac{1}{h^n} \sum_{i=0}^{n} A_i f(x + a_i h), \end{aligned} \quad (1.2.2)$$

where

$$A_i = \frac{n!}{\prod_{\substack{j=0 \\ j \ne i}}^{n} (a_i - a_j)}. \quad (1.2.3)$$

By (1.1.3) and (1.1.11)

$$\begin{aligned}\sum_{i=0}^{n} A_i a_i^p &= n! \sum_{i=0}^{n} \frac{a_i^p}{\prod_{\substack{j=0\\ j\neq i}}^{n}(a_i-a_j)} = n! Q_n(x^p; a_0, \ldots, a_n) \qquad (1.2.4)\\ &= \begin{cases} 0, & \text{if } p = 0, \ldots, n-1,\\ n!, & \text{if } p = n.\end{cases}\end{aligned}$$

Special Cases **Case I** Let $a_i = i$, $1 = 0, \ldots, n$. Then

$$\begin{aligned}\prod_{\substack{j=0\\ j\neq i}}^{n}(a_i-a_j) &= \prod_{\substack{j=0\\ j\neq i}}^{n}(i-j)\\ &= i(i-1)\cdots 1(-1)(-2)\cdots(i-n)\\ &= (-1)^{(n-i)} i!(n-i)! \qquad (1.2.5)\end{aligned}$$

and so

$$A_i = \frac{n!}{(-1)^{(n-i)} i!(n-i)!} = (-1)^{(n-i)}\binom{n}{i}.$$

In this case, we get the *unsymmetric Riemann derivative of f at x of order n:*

$$RD_n f(x) = \lim_{h\to 0}\frac{1}{h^n}\sum_{i=0}^{n} A_i f(x+a_i h) = \lim_{h\to 0}\frac{1}{h^n}\sum_{i=0}^{n}(-1)^{(n-i)}\binom{n}{i} f(x+ih). \qquad (1.2.6)$$

For $n = 1$, this is

$$RD_1 f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}, \qquad (1.2.7)$$

the ordinary derivative of f at x.
For $n = 2$, we get

$$RD_2 f(x) = \lim_{h\to 0}\frac{f(x+2h)-2f(x+h)+f(x)}{h^2}. \qquad (1.2.8)$$

Example. If $f(x) = |x|$, then $RD_2 f(x) = 0$ for all x, although $RD_1 f(0)$ does not exist. So, if the nth derivative $RD_n f(x)$ exists, the previous derivatives may not exist.

Case II. Let $a_i = 2i - n$, $i = 0, 1, \ldots, n$. Then, as in (1.2.5),

$$\begin{aligned}\prod_{\substack{j=0\\ j\neq i}}^{n}(a_i-a_j) &= \prod_{\substack{j=0\\ j\neq i}}^{n}(2i-2j) = 2^n\prod_{\substack{j=0\\ j\neq i}}^{n}(i-j)\\ &= 2^n(-1)^{n-i} i!(n-i)! \qquad (1.2.9)\end{aligned}$$

and thus

$$A_i = \frac{n!}{2^n(-1)^{n-i} i!(n-i)!} = \frac{1}{2^n}(-1)^{n-i}\binom{n}{i}. \qquad (1.2.10)$$

In this case, we get the *symmetric Riemann derivative of f at x of order n:*

$$\begin{aligned} RD_n^s f(x) &= \lim_{h\to 0} \frac{1}{h^n} \sum_{i=0}^{n} A_i f(x + a_i h) \qquad (1.2.11)\\ &= \lim_{h\to 0} \frac{1}{(2h)^n} \sum_{i=0}^{n} (-1)^{(n-i)} \binom{n}{i} f(x + 2ih - nh). \end{aligned}$$

For $n = 1$, this is

$$RD_1^s f(x) = \lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h}, \qquad (1.2.12)$$

which is the *first symmetric derivative of f at x.*
For $n = 2$, this is

$$RD_2^s f(x) = \lim_{h\to 0} \frac{f(x+2h) - 2f(x) + f(x-2h)}{(2h)^2}, \qquad (1.2.13)$$

which is the *second symmetric derivative of f at x.*

Example. If $f(x) = |x|$, then $RD_1^s f(x) = 1$ for $x > 0$, $RD_1^s f(x) = -1$ for $x < 0$, $RD_1^s f(0) = 0$ while $RD_2^s f(x) = 0$ for $x \neq 0$, and $RD_2^s f(0) = \infty$.

If the limits in (1.2.1) and consequently in (1.2.6) and (1.2.11) do not exist then the corresponding upper and lower limits are denoted by $\overline{GD}_n f(x;B), \underline{GD}_n f(x;B), \overline{RD}_n f(x), \underline{RD}_n f(x), \overline{RD}_n^s f(x), \underline{RD}_n^s f(x)$, respectively. The unilateral notion of these are obtained by suitably restricting h while taking the limits.

The derivatives $RD_n f(x)$ and $RD_n^s f(x)$ also can be defined with the help of difference operators.

For $RD_n f(x)$, consider the operator, defined by induction

$$\Delta_1(f, x, h) = f(x+h) - f(x) \qquad (1.2.14)$$
$$\Delta_k(f, x, h) = \Delta_{k-1}(f, x+h, h) - \Delta_{k-1}(f, x, h),\ k = 2, 3, \cdots .$$

Then,

$$\Delta_n(f, x, h) = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} f(x + ih),\ n = 1, 2, 3, \cdots . \qquad (1.2.15)$$

sample content of Higher Order Derivatives (Monographs and Surveys in Pure and Applied Mathematics)

- download online Microsoft Exchange Server PowerShell Cookbook (3rd Edition) pdf, azw (kindle), epub, doc, mobi
- **click Listen to the Squawking Chicken: When Mother Knows Best, What's a Daughter To Do? A Memoir (Sort Of) pdf, azw (kindle), epub, doc, mobi**
- download online No Future: Queer Theory and the Death Drive (Series Q) pdf
- Stuff [UK] (May 2012) pdf, azw (kindle)

- http://nexson.arzamaszev.com/library/Microsoft-Exchange-Server-PowerShell-Cookbook--3rd-Edition-.pdf
- http://paulczajak.com/?library/Stalin--New-Biography-of-a-Dictator.pdf
- http://anvilpr.com/library/No-Future--Queer-Theory-and-the-Death-Drive--Series-Q-.pdf
- http://xn--d1aboelcb1f.xn--p1ai/lib/Stuff--UK---May-2012-.pdf