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Graphs and Matrices

Second Edition

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Preface

This book is concerned with results in graph theory in which linear algebra and matrix theory play an important role. Although it is generally accepted that linear algebra can be an important component in the study of graphs, traditionally, graph theorists have remained by and large less than enthusiastic about using linear algebra. The results discussed here are usually treated under *algebraic graph theory*, as outlined in the classic books by Biggs [20] and by Godsil and Royle [39]. Our emphasis on matrix techniques is even greater than what is found in these and perhaps the subject matter discussed here might be termed *linear algebraic graph theory* to highlight this aspect.

After recalling some matrix preliminaries in the Chap. 1, the next few chapters outline the basic properties of some matrices associated with a graph. This is followed by topics in graph theory such as regular graphs and algebraic connectivity. Distance matrix of a tree and its generalized version for arbitrary graphs, the resistance matrix, are treated in the next two chapters. The final chapters treat other topics such as the Laplacian eigenvalues of threshold graphs, the positive definite completion problem, and matrix games based on a graph.

We have kept the treatment at a fairly elementary level and resisted the temptation of presenting up-to-date research work. Thus, several chapters in this book may be viewed as an invitation to a vast area of vigorous current research. Only a beginning is made here with the hope that it will entice the reader to explore further. In the same vein, we often do not present the results in their full generality, but present only a simpler version that captures the elegance of the result. Weighted graphs are avoided, although most results presented here have weighted, and hence more general, analogs.

The references for each chapter are listed at the end of the chapter. In addition, a master bibliography is included. In a short note at the end of each chapter, we indicate the primary references that we used. Often, we have given a different treatment, as well as different proofs, of the results cited. We do not go into an elaborate description of such differences.

It is a pleasure to thank Rajendra Bhatia for his diligent handling of the manuscript. Aloke Dey, Arbind Lal, Sukanta Pati, Sharad Sane, S. Sivaramakrishnan, and Murali Srinivasan read either all or parts of the manuscript, suggested changes and pointed out corrections. I sincerely thank them all. Thanks are also due to the anonymous referees for helpful comments. Needless to say I remain responsible for the shortcomings and errors that persist. The facilities provided by the Indian Statistical Institute, New Delhi, and the support of the JC Bose Fellowship, Department of Science and Technology, Government of India, are gratefully acknowledged.

New Delhi, India Ravindra B. Bapat

About the Second Edition

In this edition, besides correcting some errors and typos in the first edition, we have added a new chapter on the line graph of a tree.

I sincerely thank Nazli Besharati, Arbind K. Lal, Ambat Vijayakumar, Anu Varghese, and Seethu Varghese for pointing out corrections in the first edition. I also thank Souvik Dhara, Ibrahim Ghorbani, and Rajesh Kannan for a careful reading of the new chapter and for helpful suggestions.

Contents

Chapter 1
Preliminaries

In this chapter we review certain basic concepts from linear algebra. We consider only real matrices. Although our treatment is self-contained, the reader is assumed to be familiar with the basic operations on matrices. Relevant concepts and results are given, although we omit proofs.

1.1 Matrices

Basic Definitions

An $m \times n$ matrix consists of mn real numbers arranged in m rows and n columns. The entry in row i and column j of the matrix A is denoted by a_{ij}. An $m \times 1$ matrix is called a column vector of order m; similarly, a $1 \times n$ matrix is a row vector of order n. An $m \times n$ matrix is called a square matrix if $m = n$.

Operations of matrix addition, scalar multiplication and matrix multiplication are basic and will not be recalled here. The transpose of the $m \times n$ matrix A is denoted by A'.

A diagonal matrix is a square matrix A such that $a_{ij} = 0, i \neq j$. We denote the diagonal matrix

$$\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

by $\mathsf{diag}(\lambda_1, \ldots, \lambda_n)$. When $\lambda_i = 1$ for all i, this matrix reduces to the identity matrix of order n, which we denote by I_n or often simply by I if the order is clear from the context. The matrix A is upper triangular if $a_{ij} = 0, i > j$. The transpose of an upper triangular matrix is lower triangular.

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Trace and Determinant

Let A be a square matrix of order n. The entries $a_{11}, \dots, a_{nn}$ are said to constitute the (main) diagonal of A. The *trace* of A is defined as

$$\text{trace} A = a_{11} + \cdots + a_{nn}.$$

It follows from this definition that if A, B are matrices such that both AB and BA are defined, then

$$\text{trace} AB = \text{trace} BA.$$

The *determinant* of an $n \times n$ matrix A, denoted by det A, is defined as

$$\det A = \sum_{\sigma} sgn(\sigma) a_{1\sigma(1)} \cdots a_{n\sigma(n)},$$

where the summation is over all permutations $\sigma(1), \dots, \sigma(n)$ of $1, \dots, n$, and $sgn(\sigma)$ is 1 or -1 according as σ is even or odd. We assume familiarity with the basic properties of determinant.

Vector Spaces Associated with a Matrix

Let $\mathbb{R}$ denote the set of real numbers. Consider the set of all column vectors of order n ($n \times 1$ matrices) and the set of all row vectors of order n ($1 \times n$ matrices). Both of these sets will be denoted by $\mathbb{R}^n$. We will write the elements of $\mathbb{R}^n$ either as column vectors or as row vectors, depending upon whichever is convenient in a given situation. Recall that $\mathbb{R}^n$ is a vector space with the operations matrix addition and scalar multiplication.

Let A be an $m \times n$ matrix. The subspace of $\mathbb{R}^m$ spanned by the column vectors of A is called the *column space* or the *column span* of A. Similarly the subspace of $\mathbb{R}^n$ spanned by the row vectors of A is called the row space of A.

According to the fundamental theorem of linear algebra, the dimension of the column space of a matrix equals the dimension of the row space, and the common value is called the *rank* of the matrix. We denote the rank of the matrix A by rank A.

For any matrix A, $\text{rank} A = \text{rank} A'$. If A and B are matrices of the same order, then $\text{rank}(A + B) \le \text{rank} A + \text{rank} B$. If A and B are matrices such that AB is defined, then $\text{rank} AB \le \min\{\text{rank} A, \text{rank} B\}$.

Let A be an $m \times n$ matrix. The set of all vectors $x \in \mathbb{R}^n$ such that $Ax = 0$ is easily seen to be a subspace of $\mathbb{R}^n$. This subspace is called the *null space* of A, and we denote it by $\mathcal{N}(A)$. The dimension of $\mathcal{N}(A)$ is called the *nullity* of A. Let A be an $m \times n$ matrix. Then the nullity of A equals $n - \text{rank} A$.

Minors

Let A be an $m \times n$ matrix. If $S \subset \{1, \ldots, m\}, T \subset \{1, \ldots, n\}$, then $A[S|T]$ will denote the submatrix of A determined by the rows corresponding to S and the columns corresponding to T. The submatrix obtained by deleting the rows in S and the columns in T will be denoted by $A(S|T)$. Thus, $A(S|T) = A[S^c|T^c]$, where the superscript c denotes complement. Often, we tacitly assume that S and T are such that these matrices are not vacuous. When $S = \{i\}, T = \{j\}$ are singletons, then $A(S|T)$ is denoted $A(i|j)$.

Nonsingular Matrices

A matrix A of order $n \times n$ is said to be *nonsingular* if $\operatorname{rank} A = n$; otherwise the matrix is *singular*. If A is nonsingular, then there is a unique $n \times n$ matrix A^{-1}, called the inverse of A, such that $AA^{-1} = A^{-1}A = I$. A matrix is nonsingular if and only if $\det A$ is nonzero.

The cofactor of a_{ij} is defined as $(-1)^{i+j} \det A(i|j)$. The adjoint of A is the $n \times n$ matrix whose (i, j)th entry is the cofactor of a_{ji}. We recall that if A is nonsingular, then A^{-1} is given by $\dfrac{1}{\det A}$ times the adjoint of A.

A matrix is said to have full column rank if its rank equals the number of columns, or equivalently, the columns are linearly independent. Similarly, a matrix has full row rank if its rows are linearly independent. If B has full column rank, then it admits a left inverse, that is, a matrix X such that $XB = I$. Similarly, if C has full row rank, then it has a right inverse, that is, a matrix Y such that $CY = I$.

If A is an $m \times n$ matrix of rank r then we can write $A = BC$, where B is $m \times r$ of full column rank and C is $r \times n$ of full row rank. This is called a *rank factorization* of A. There exist nonsingular matrices P and Q of order $m \times m$ and $n \times n$, respectively, such that

$$A = P \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q.$$

This is the rank canonical form of A.

Orthogonality

Vectors x, y in $\mathbb{R}^n$ are said to be orthogonal, or perpendicular, if $x'y = 0$. A set of vectors $\{x_1, \ldots, x_m\}$ in $\mathbb{R}^n$ is said to form an *orthonormal basis* for the vector space S if the set is a basis for S, and furthermore $x_i'x_j$ is 0 if $i \neq j$, and 1 if $i = j$. The $n \times n$ matrix P is said to be orthogonal if $PP' = P'P = I$. One can verify that if P is orthogonal then P' is orthogonal.

If $x_1, \dots, x_k$ are linearly independent vectors then by the Gram–Schmidt orthogonalization process we may construct orthonormal vectors $y_1, \dots, y_k$ such that y_i is a linear combination of $x_1, \dots, x_i$; $i = 1, \dots, k$.

Schur Complement

Let A be an $n \times n$ matrix partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{1.1}$$

where A_{11} and A_{22} are square matrices. If A_{11} is nonsingular then the *Schur complement* of A_{11} in A is defined to be the matrix $A_{22} - A_{21}A_{11}^{-1}A_{12}$. Similarly, if A_{22} is nonsingular then the Schur complement of A_{22} in A is $A_{11} - A_{12}A_{22}^{-1}A_{21}$.

The following identity is easily verified:

$$\begin{bmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} I & -A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix}. \tag{1.2}$$

The following useful fact can be easily proved using (1.2):

$$\det A = (\det A_{11}) \det(A_{22} - A_{21}A_{11}^{-1}A_{12}). \tag{1.3}$$

We will refer to (1.3) as the Schur complement formula, or the Schur formula, for the determinant.

Inverse of a Partitioned Matrix

Let A be an $n \times n$ nonsingular matrix partitioned as in (1.1). Suppose A_{11} is square and nonsingular and let $A/A_{11} = A_{22} - A_{21}A_{11}^{-1}A_{12}$ be the Schur complement of A_{11}. Then

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}(A/A_{11})^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}(A/A_{11})^{-1} \\ -(A/A_{11})^{-1}A_{21}A_{11}^{-1} & (A/A_{11})^{-1} \end{bmatrix}.$$

Note that if A and A_{11} are nonsingular, then A/A_{11} must be nonsingular. Equivalent formulae may be given in terms of the Schur complement of A_{22}.

Cauchy–Binet Formula

Let A and B be matrices of order $m \times n$ and $n \times m$ respectively, where $m \leq n$. Then

$$\det(AB) = \sum \det A[\{1, \ldots, m\}|S] \det B[S|\{1, \ldots, m\}],$$

where the summation is over all m-element subsets of $\{1, \ldots, n\}$.

To illustrate by an example, let

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 \\ 0 & 3 \\ 5 & 1 \end{bmatrix}.$$

Then $\det(AB)$ equals

$$\det \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} + \det \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 5 & 1 \end{bmatrix} + \det \begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 5 & 1 \end{bmatrix}.$$

1.2 Eigenvalues of Symmetric Matrices

Characteristic Polynomial

Let A be an $n \times n$ matrix. The determinant $\det(A - \lambda I)$ is a polynomial in the (complex) variable λ of degree n and is called the *characteristic polynomial* of A. The equation

$$\det(A - \lambda I) = 0$$

is called the *characteristic equation* of A. By the fundamental theorem of algebra the equation has n complex roots and these roots are called the *eigenvalues* of A.

We remark that it is customary to define the characteristic polynomial of A as $\det(\lambda I - A)$ as well. This does not affect the eigenvalues.

The eigenvalues might not all be distinct. The number of times an eigenvalue occurs as a root of the characteristic equation is called the *algebraic multiplicity* of the eigenvalue.

We may factor the characteristic polynomial as

$$\det(A - \lambda I) = (\lambda_1 - \lambda) \cdots (\lambda_n - \lambda).$$

The geometric multiplicity of the eigenvalue λ of A is defined to be the dimension of the null space of $A - \lambda I$. The geometric multiplicity of an eigenvalue does not exceed its algebraic multiplicity.

If A and B are matrices of order $m \times n$ and $n \times m$, respectively, where $m \geq n$, then the eigenvalues of AB are the same as the eigenvalues of BA, along with 0 with a (possibly further) multiplicity of $m - n$.

If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A, then $\det A = \lambda_1 \cdots \lambda_n$, while $\mathit{trace}\, A = \lambda_1 + \cdots + \lambda_n$.

A principal submatrix of a square matrix is a submatrix formed by a set of rows and the corresponding set of columns. A principal minor of A is the determinant of a principal submatrix. A leading principal minor is a principal minor involving rows and columns $1, \ldots, k$ for some k.

The sum of the products of the eigenvalues, of A, taken k at a time, equals the sum of the $k \times k$ principal minors of A. When $k = 1$ this reduces to the familiar fact that the sum of the eigenvalues equals the trace.

If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the $n \times n$ matrix A, and if $q(A)$ is a polynomial in A, then the eigenvalues of $q(A)$ are $q(\lambda_1), \ldots, q(\lambda_n)$.

If A is an $n \times n$ matrix with the characteristic polynomial $p(A)$, then the Cayley–Hamilton theorem asserts that $p(A) = 0$. The monic polynomial $q(x)$ of minimum degree that satisfies $q(A) = 0$ is called the *minimal polynomial* of A.

Spectral Theorem

A square matrix A is called *symmetric* if $A = A'$. The eigenvalues of a symmetric matrix are real. Furthermore, if A is a symmetric $n \times n$ matrix, then according to the spectral theorem there exists an orthogonal matrix P such that

$$PAP' = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

In the case of a symmetric matrix the algebraic and the geometric multiplicities of any eigenvalue coincide. Also, the rank of the matrix equals the number of nonzero eigenvalues, counting multiplicities.

Let A and B be symmetric $n \times n$ matrices such that they commute, i.e., $AB = BA$. Then A and B can be simultaneously diagonalized, that is, there exists an orthogonal matrix P such that PAP' and PBP' are both diagonal, with the eigenvalues of A (respectively, B) along the diagonal PAP' (respectively, PBP').

Positive Definite Matrices

An $n \times n$ matrix A is said to be *positive definite* if it is symmetric and if for any nonzero vector x, $x'Ax > 0$. The identity matrix is clearly positive definite and so is a diagonal matrix with only positive entries along the diagonal. Let A be a symmetric $n \times n$ matrix. Then any of the following conditions is equivalent to A being positive definite:

(i) the eigenvalues of A are positive;
(ii) all principal minors of A are positive;
(iii) all leading principal minors of A are positive;
(iv) $A = BB'$ for some matrix B of full column rank;
(v) $A = TT'$ for some lower triangular matrix T with positive diagonal entries.

A symmetric matrix A is called *positive semidefinite* if $x'Ax \geq 0$ for any x. Equivalent conditions for a matrix to be positive semidefinite can be given similarly. However, note that the leading principal minors of A may be nonnegative and yet A may not be positive semidefinite. This is illustrated by the example $\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$. Also, in (v), the diagonal entries of T need only be nonnegative.

If A is positive semidefinite then there exists a unique positive semidefinite matrix B such that $B^2 = A$. The matrix B is called the *square root* of A and is denoted by $A^{1/2}$.

Let A be an $n \times n$ matrix partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{1.4}$$

where A_{11} and A_{22} are square matrices.

The following facts can be easily proved using (1.2):

(i) If A is positive definite then $A_{22} - A_{21}A_{11}^{-1}A_{12}$ is positive definite;
(ii) Let A be symmetric. If A_{11} and its Schur complement $A_{22} - A_{21}A_{11}^{-1}A_{12}$ are both positive definite then A is positive definite.

Interlacing for Eigenvalues

The following result, known as the Cauchy interlacing theorem, finds considerable use in graph theory.

Let A be a symmetric $n \times n$ matrix and let B be a principal submatrix of A of order $n-1$. If $\lambda_1 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \cdots \geq \mu_{n-1}$ are the eigenvalues of A and B, respectively, then

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n. \tag{1.5}$$

A related interlacing result is as follows. Let A and B be symmetric $n \times n$ matrices and let $A = B + xx'$ for some vector x. If $\lambda_1 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \cdots \geq \mu_n$ are the eigenvalues of A and B respectively, then

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq \mu_n. \tag{1.6}$$

Let A be a symmetric $n \times n$ matrix with eigenvalues $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$, arranged in nonincreasing order. Let $||x||$ denote the usual Euclidean norm, $\left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}$. The following extremal representation will be useful:

$$\lambda_1(A) = \max_{||x||=1} \{x'Ax\}, \quad \lambda_n(A) = \min_{||x||=1} \{x'Ax\}.$$

Setting x to be the ith column of I in the above representation we see that

$$\lambda_n(A) \leq \min_i \{a_{ii}\} \leq \max_i \{a_{ii}\} \leq \lambda_1(A).$$

1.3 Generalized Inverses

Let A be an $m \times n$ matrix. A matrix G of order $n \times m$ is said to be a *generalized inverse* (or a g-*inverse*) of A if $AGA = A$. If A is square and nonsingular then A^{-1} is the unique g-inverse of A. Otherwise, A has infinitely many g-inverses, as we will see shortly.

Let A be an $m \times n$ matrix and let G be a g-inverse of A. If $Ax = b$ is consistent then $x = Gb$ is a solution of $Ax = b$.

Let $A = BC$ be a rank factorization. Then B admits a left inverse B_ℓ^- and C admits a right inverse C_r^-. Then $G = C_r^- B_\ell^-$ is a g-inverse of A, since

$$AGA = BC(C_r^- B_\ell^-)BC = BC = A.$$

Alternatively, if A has rank r then there exist nonsingular matrices P, Q such that

$$A = P \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q.$$

It can be verified that for any U, V, W of appropriate dimensions,

$$\begin{bmatrix} I_r & U \\ V & W \end{bmatrix}$$

is a g-inverse of

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

Then

$$G = Q^{-1} \begin{bmatrix} I_r & U \\ V & W \end{bmatrix} P^{-1}$$

is a g-inverse of A. This also shows that any matrix that is not a square, nonsingular matrix admits infinitely many g-inverses.

Another method that is particularly suitable for computing a g-inverse is as follows. Let A be of rank r. Choose any $r \times r$ nonsingular submatrix of A. For convenience let us assume

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where A_{11} is $r \times r$ and nonsingular. Since A has rank r, there exists a matrix X such that $A_{12} = A_{11}X$, $A_{22} = A_{21}X$. Now it can be verified that the $n \times m$ matrix G defined as

$$G = \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

is a g-inverse of A. (Just multiply AGA out to see this.) We will often use the notation A^- to denote a g-inverse of A.

A g-inverse of A is called a *reflexive* g-*inverse* if it also satisfies $GAG = G$. Observe that if G is any g-inverse of A then GAG is a reflexive g-inverse of A.

Let A be an $m \times n$ matrix, G be a g-inverse of A and y be in the column space of A. Then the class of solutions of $Ax = y$ is given by $Gy + (I - GA)z$, where z is arbitrary.

A g-inverse G of A is said to be a *minimum norm* g-inverse of A if, in addition to $AGA = A$, it satisfies $(GA)' = GA$. If G is a minimum norm g-inverse of A, then for any y in the column space of A, $x = Gy$ is a solution of $Ax = y$ with minimum norm. A proof of this fact will be given in Chap. 9.

A g-inverse G of A is said to be a *least squares* g-inverse of A if, in addition to $AGA = A$, it satisfies $(AG)' = AG$. If G is a least squares g-inverse of A then for any x, y, $||AGy - y|| \leq ||Ax - y||$.

Moore–Penrose Inverse

If G is a reflexive g-inverse of A that is both minimum norm and least squares then it is called a *Moore–Penrose inverse* of A. In other words, G is a Moore–Penrose inverse of A if it satisfies

$$AGA = A, \quad GAG = G, \quad (AG)' = AG, \quad (GA)' = GA. \tag{1.7}$$

We will show that such a G exists and is, in fact, unique. We first show uniqueness. Suppose G_1, G_2 both satisfy (1.7). Then we must show $G_1 = G_2$. The derivation is as follows.

$$\begin{aligned}G_1 &= G_1AG_1 = G_1G_1'A' = G_1G_1'A'G_2'A' = G_1G_1'A'AG_2\\&= G_1AG_1AG_2 = G_1AG_2 = G_1AG_2AG_2 = G_1AA'G_2'G_2\\&= A'G_1'A'G_2'G_2 = A'G_2'G_2 = G_2AG_2 = G_2.\end{aligned}$$

We will denote the Moore–Penrose inverse of A by A^+. We now show the existence. Let $A = BC$ be a rank factorization. Then it can be easily verified that

$$B^+ = (B'B)^{-1}B', \quad C^+ = C'(CC')^{-1}$$

and then

$$A^+ = C^+B^+.$$

Let A be a symmetric $n \times n$ matrix and let P be an orthogonal matrix such that

$$A = P \operatorname{diag}(\lambda_1, \dots, \lambda_n)P'.$$

If $\lambda_1, \dots, \lambda_r$ are the nonzero eigenvalues then

$$A^+ = P \operatorname{diag}\left(\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_r}, 0, \dots, 0\right) P'.$$

In particular, if A is positive semidefinite, then so is A^+.

1.4 Graphs

We assume familiarity with basic theory of graphs. A graph G consists of a finite set of vertices $V(G)$ and a set of edges $E(G)$ consisting of distinct, unordered pairs of vertices. We usually take $V(G)$ to be $\{1, \dots, n\}$ and $E(G)$ to be $\{e_1, \dots, e_m\}$. We may refer to edges $j_1, j_2, \dots$ when we actually mean edges $e_{j_1}, e_{j_2}, \dots$. We consider simple graphs, that is, graphs without loops and parallel edges. Our emphasis is on undirected graphs. However, we do consider directed graphs as well.

If e_k is an edge with end-vertices i and j, then we say that e_k and i or e_k and j are incident. We also write $e_k = \{i, j\}$. The notation $i \sim j$ is used to indicate that i and j are joined by an edge, or that they are adjacent.

Notions such as connected graph, subgraph, degree, path, cycle and so on are standard and will not be recalled here. The complement of the graph G will be denoted by G^c. The complete graph on n vertices will be denoted by K_n. The complete bipartite graph with partite sets of cardinality m, n, will be denoted by $K_{m,n}$. Note that $K_{1,n}$ is called a *star*. Further notions will be recalled as and when the need arises.

Exercises

1. Let A be an $m \times n$ matrix. Show that A and $A'A$ have the same null space. Hence conclude that $\operatorname{rank} A = \operatorname{rank} A'A$.
2. Let A be a matrix in partitioned form:

$$A = \begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ A_{21} & A_{22} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ A_{k1} & A_{k2} & \cdots & A_{kk} \end{bmatrix}.$$

   Show that $\operatorname{rank} A \geq \operatorname{rank} A_{11} + \cdots + \operatorname{rank} A_{kk}$, and that equality holds if $A_{ij} = 0$, $i > j$.
3. Let A be an orthogonal $n \times n$ matrix. Show that a_{11} and $\det A(1|1)$ have the same absolute value.
4. Let A and G be matrices of order $m \times n$ and $n \times m$, respectively. Show that $G = A^+$ if and only if $A'AG = A'$ and $G'GA = G'$.
5. If A is a matrix of rank 1, then show that $A^+ = \alpha A'$ for some α. Determine α.

It would be difficult to list the many excellent books that provide the necessary background outlined in this chapter. A few selected references are indicated below.

The books [Bap00] and [HJ85] contain the required matrix theory preliminaries, while [BM08] and [Wes02] are standard introductions to graph theory. The books [BG03] and [CM79] are comprehensive references on generalized inverses.

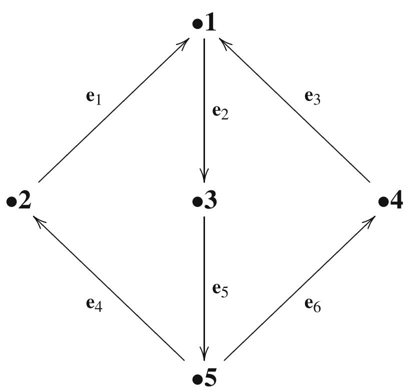
References and Further Reading

[Bap00] Bapat, R.B.: Linear Algebra and Linear Models, 2nd edn. Hindustan Book Agency, New Delhi, and Springer, Heidelberg (2000)
[BG03] Ben-Israel, A., Greville, T.N.E.: Generalized Inverses: Theory and Applications, 2nd edn. Springer, New York (2003)
[BM08] Bondy, J.A., Murty, U.S.R.: Graph Theory, Graduate Texts in Mathematics, 244. Springer, New York (2008)
[CM79] Campbell, S.L., Meyer, C.D.: Generalized Inverses of Linear Transformation. Pitman, London (1979)
[HJ85] Horn, R.A., Johnson, C.R.: Matrix Analysis. Cambridge University Press, Cambridge (1985)
[Wes02] West, D.: Introduction to Graph Theory, 2nd edn. Prentice-Hall, New Delhi (2002)

Chapter 2
Incidence Matrix

Let G be a graph with $V(G) = \{1, \dots, n\}$ and $E(G) = \{e_1, \dots, e_m\}$. Suppose each edge of G is assigned an orientation, which is arbitrary but fixed. The (*vertex-edge*) *incidence matrix* of G, denoted by $Q(G)$, is the $n \times m$ matrix defined as follows. The rows and the columns of $Q(G)$ are indexed by $V(G)$ and $E(G)$, respectively. The (i, j)-entry of $Q(G)$ is 0 if vertex i and edge e_j are not incident, and otherwise it is 1 or -1 according as e_j originates or terminates at i, respectively. We often denote $Q(G)$ simply by Q. Whenever we mention $Q(G)$ it is assumed that the edges of G are oriented.

Example 2.1 Consider the graph shown. Its incidence matrix is given by Q.



$$Q = \begin{bmatrix} -1 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{bmatrix}$$

R.B. Bapat, *Graphs and Matrices*, Universitext,
DOI 10.1007/978-1-4471-6569-9_2

2.1 Rank

For any graph G, the column sums of $Q(G)$ are zero and hence the rows of $Q(G)$ are linearly dependent. We now proceed to determine the rank of $Q(G)$.

Lemma 2.2 *If G is a connected graph on n vertices, then* rank $Q(G) = n - 1$.

Proof Suppose x is a vector in the left null space of $Q := Q(G)$, that is, $x'Q = 0$. Then $x_i - x_j = 0$ whenever $i \sim j$. It follows that $x_i = x_j$ whenever there is an ij-path. Since G is connected, x must have all components equal. Thus, the left null space of Q is at most one-dimensional and therefore the rank of Q is at least $n - 1$. Also, as observed earlier, the rows of Q are linearly dependent and therefore rank $Q \leq n - 1$. Hence, rank $Q = n - 1$. □

Theorem 2.3 *If G is a graph on n vertices and has k connected components then* rank $Q(G) = n - k$.

Proof Let $G_1, \ldots, G_k$ be the connected components of G. Then, after a relabeling of vertices (rows) and edges (columns) if necessary, we have

$$Q(G) = \begin{bmatrix} Q(G_1) & 0 & \cdots & 0 \\ 0 & Q(G_2) & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & Q(G_k) \end{bmatrix}.$$

Since G_i is connected, rank $Q(G_i)$ is $n_i - 1$, where n_i is the number of vertices in G_i, $i = 1, \ldots, k$. It follows that

$$\begin{aligned} \operatorname{rank} Q(G) &= \operatorname{rank} Q(G_1) + \cdots + \operatorname{rank} Q(G_k) \\ &= (n_1 - 1) + \cdots + (n_k - 1) \\ &= n_1 + \cdots + n_k - k = n - k. \end{aligned}$$

This completes the proof.

Lemma 2.4 *Let G be a connected graph on n vertices. Then the column space of $Q(G)$ consists of all vectors $x \in \mathbb{R}^n$ such that $\sum_i x_i = 0$.*

Proof Let U be the column space of $Q(G)$ and let

$$W = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 0 \right\}.$$

Then dim $W = n - 1$. Each column of $Q(G)$ is clearly in W and hence $U \subset W$. It follows by Lemma 2.2 that