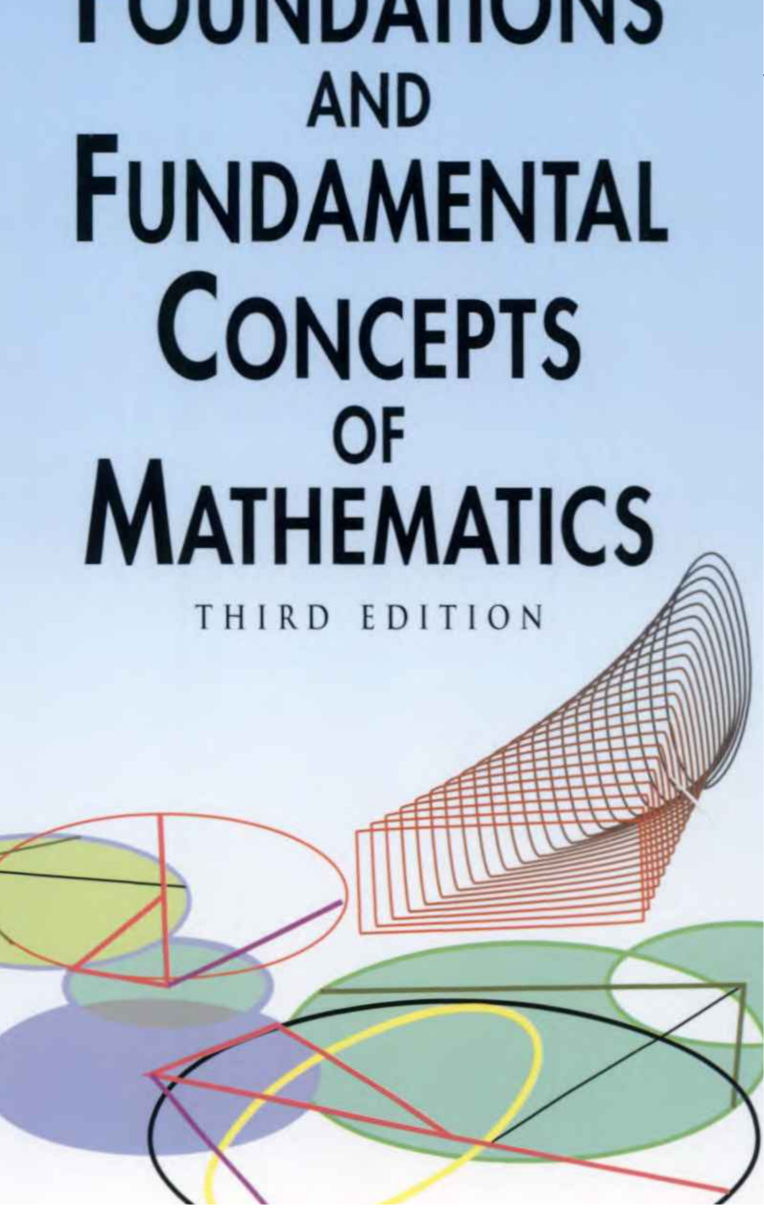
FOUNDATIONS
AND
FUNDAMENTAL
CONCEPTS
OF
MATHEMATICS
THIRD EDITION

DOVER BOOKS ON MATHEMATICS

(continued on back flap)

FOUNDATIONS AND FUNDAMENTAL CONCEPTS OF MATHEMATICS

THIRD EDITION

Howard Eves
University of Maine
University of Central Florida

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To the Memory of Pride and Kabar
Two Great Friends and Companions

FOREWORD

Mathematical procedures are widely recognized as increasingly necessary and powerful tools in scientific explorations, industrial developments, and many of our personal activities in this age of scientific and technological innovations. This third edition of *Foundations and Fundamental Concepts of Mathematics* provides a welcome opportunity for college undergraduates to obtain an overview of the historical roots and the evolution of several areas of mathematics.

The selection of topics conveys not only their role in this historical development of mathematics but also their value as bases for understanding the changing nature of mathematics. The continuing rapid growth of mathematics makes it impossible to cover all aspects of the subject in a single course or year of courses. Indeed, Henri Poincaré (1854–1912) is frequently cited as the last universal mathematician, the last person who understood all of the aspects of mathematics that were known in his or her time. Our present situation is further complicated by the fact that the scope of the mathematical sciences has at least doubled during the last seventy-five years.

The topics included in *Foundations and Fundamental Concepts of Mathematics*, Third Edition, have special significance for mathematics majors and prospective teachers of mathematics. In particular, the emphasis on axiomatic procedures provides an important background for studying and applying more advanced topics. The inclusion of the historical roots of both algebra and geometry provides an essential background for prospective teachers of school mathematics.

The readable style and sets of challenging exercises from the popular earlier editions have been continued and extended in this third edition, making it a very welcome and useful version of a classic treatment of the foundations of mathematics. It is always a happy occasion to welcome the reappearance in revised form of a truly satisfying book.

Bruce E. Meserve
Professor Emeritus, *University of Vermont*

PREFACE

The story of the development of mathematics is made up of two intertwined strands. One strand narrates the growing content of mathematics and the other the changing nature of mathematics. Almost everyone realizes that mathematics must have arisen from some very slender beginnings far back in time and then gradually grown into its present enormous structure, but not so many realize that the nature and very meaning of mathematics have also changed and evolved over the ages. The story of the growing content of mathematics constitutes the subject matter of books and courses on the history of mathematics, and the story of the changing nature of mathematics constitutes the subject matter of books and courses devoted to the foundations, philosophy, and fundamental concepts of mathematics. For a properly rounded picture of the development of mathematics, both stories (which are highly interlinked) should be studied, and each should constitute a "must" course for all serious students of mathematics and all prospective teachers of mathematics.

Accordingly, two companion volumes were conceived, one to be devoted to the history of mathematics and one to the foundations and fundamental concepts of mathematics. The task was begun under the enthusiastic encouragement of Carroll V. Newsom, then (among many other things) the very able mathematics editor of Rinehart Book Company. The unavoidable bits of overlap of some of the topics of the two books was not considered as constituting any great sin of duplication, for these overlaps would be viewed somewhat differently in the two treatments. In each case, the presentation was to be on a level understandable by the able undergraduate college mathematics student. The first of the two books to be written was the *History* book, followed a few years later by the *Foundations* book. Thus started a fine association with Rinehart Book Company, in those wonderful days when the author–publisher relationship was a true partnership.

Unfortunately, the intimate union of the two books as companion volumes was fated not to continue. Rinehart was swallowed by Holt (to become Holt, Rinehart and Winston), and the two books somehow lost their twinship; and when Holt decided to give up its mathematics offerings, the *History* book was given up for adoption by Saunders College Publishing and the *Foundations* book was allowed to expire. Thus the intended union of the two books came to an end and the expired volume lay dead for a number of years, in spite of a wide wish of many of the mathematical fraternity that it be brought back to life. The resurrection was finally instigated by Steve Quigley, senior mathematics editor of PWS-KENT Publishing Company.

It is perhaps pertinent to quote here, with some slight amendment, a few paragraphs from the preface to the original edition of the *Foundations* book.

> There is little doubt that man's rapid progress in recent decades in the control and understanding of nature, in providing himself abstract tools of creation that border upon the miraculous, and in his actual comprehension of the powers and limitations of the human mind is a direct consequence of mathematical triumphs of the last few centuries, especially the nineteenth and twentieth. Thus scholarly endeavor in virtually all areas of human activity requires to an increasing extent a knowledge of mathematics and the ability to use it; except for the study of language, mathematics may well be the most basic component of a so-called general education.
>
> Yet even students of mathematics are usually denied until their most advanced years of study an understanding of the meaning and nature of mathematics. They labor under false definitions and impressions, which, unfortunately, continue to be promulgated in most elementary courses. Until they have reached the graduate level, students generally have heard little of mathematical structure; they have virtually no acquaintance with the common collections of axioms, with the postulational method, and with the nature and use of mathematical systems and models; it is probable that they have little genuine knowledge of the real number system and have had little more than a superficial experience in working with such a fundamental concept as *set*.
>
> This book has been written, therefore, in an attempt to make available to able undergraduate students an introductory treatment of the foundations of mathematics. A course of study utilizing this work as a text would, it is hoped, rectify a great deal of the curricular deficiency just described. The contents were selected with considerable care in order that the exposition might have value not only to mathematical scholars and scientists but also to philosophers, historians, and others; it is especially hoped that potential teachers of mathematics will have an opportunity to study under competent instruction such material as this book contains.
>
> The treatment is strongly historical, for the study is concerned with fundamental mathematical ideas; a genuine understanding of ideas is not possible without an analysis of origins. Even the order of topics, as revealed by the table of contents, provides in a rough way a chronological development of the basic concepts that have made mathematics what it is today. Obviously, the treatment at many points is far from exhaustive, for the exposition has been designed to be compatible with the level of maturity and understanding of the able undergraduate student; the bibliography contains many suggestions for those who desire to learn more about a particular topic.
>
> The book will have accomplished its purpose for many readers if it does no more for them than explain the nature of geometric and algebraic systems. It is hoped, however, that much more will be accomplished.

Perhaps some students of mathematics will for the first time see the forest without becoming confused by the trees; and a few students may obtain from the work a glimpse of the meaning and opportunities of mathematical creation, the finest testimony to humankind's inherent genius.

Finally, it is a pleasure to repeat former thanks to The Cambridge University Press, The Macmillan Company, McGraw-Hill Book Company Inc., Mrs. R. E. Moritz and Mrs. C. J. Keyser for graciously granting permission to quote from certain works. Much of the task of writing the book was made easier by the understanding and cooperation of Professor Spofford H. Kimball, at the time chairman of the Department of Mathematics at the University of Maine. A large part of the manuscript was read and critically discussed from the philosophical point of view by Dr. Charles G. Werner, then of the University of Miami. An undoubted debt is also owed to Professor Raymond L. Wilder, with whom Carroll Newsom first studied such matters as are contained in the book, and to Professor Clayton W. Dodge of the University of Maine for his invaluable counsel in the preparation of the second edition of the work. And, of course, special thanks go to Steve Quigley for his energetic efforts in bringing the book back to life.

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1

MATHEMATICS BEFORE EUCLID

1.1 The Empirical Nature of Pre-Hellenic Mathematics

The thesis can be advanced that mathematics arose from necessity. The annual inundation of the Nile Valley, for example, forced the Egyptians to develop some system for redetermining land markings; in fact, the word *geometry* means "measurement of the earth." The need for mensuration formulas was especially imperative if, as Herodotus remarked, taxes in Egypt were paid on the basis of land area. The Babylonians likewise encountered an urgent need for mathematics in the construction of the great engineering structures for which they were famous. Marsh drainage, irrigation, and flood control made it possible to convert the land along the Tigris and Euphrates rivers into a rich agricultural region. Similar undertakings undoubtedly occurred in early times in south-central Asia along the Indus and Ganges rivers, and in eastern Asia along the Hwang Ho and the Yangtze. The engineering, financing, and administration of such projects required the development of considerable technical knowledge and its attendant mathematics. A useable calendar had to be computed to serve agricultural needs, and this required some basic astronomy with its concomitant mathematics. Again, the demand for some system of uniformity in barter was present in even the earliest civilizations; this fact also furnished a pronounced stimulus to mathematical development. Finally, early religious ritual found need for some basic mathematics.[1]

Thus there is a basis for saying that mathematics, beyond that implied by primitive counting, originated during the period of the fifth, fourth, and third millennia B.C. in certain areas of the ancient orient as a practical science to assist in engineering, agricultural, and business pursuits and in religious ritual. Although the initial emphasis was on mensuration and practical arithmetic, it

[1] See A. Seidenberg [1] and [2]. (References by author's name only are to the Bibliography at the end of the book.)

was natural that a special craft should come into being for the application, instruction, and development of the science and that, in turn, tendencies toward abstraction should then assert themselves and the subject be studied, to some extent, for its own sake. In this way a basis for the beginnings of theoretical geometry grew out of mensuration, and the first traces of elementary algebra evolved from practical arithmetic.[2]

In our study of early mathematics we are restricted essentially to that of Egypt and Babylonia. The ancient Egyptians recorded their work on stone and papyrus, the latter fortunately enduring because of Egypt's unusually dry climate; the Babylonians used imperishable baked clay tablets. In contrast to the use of these media, the early Indians and Chinese used very perishable writing materials like bark and bamboo. Thus it has come to pass that we have a fair quantity of definite information, obtained from primary sources, about the science and the mathematics of ancient Egypt and Babylonia, while we know very little indeed, with any degree of certainty, about these fields of study in ancient India and China.

It is the nature, rather than the content, of this pre-Hellenic mathematics that concerns us here, and in this regard it is important to note that, outside of very simple considerations, the mathematical relations employed by the Egyptians and by the Babylonians resulted essentially from "trial and error" methods. In other words, to a great extent the earliest mathematics was little more than a practically workable empiricism—a collection of rule-of-thumb procedures that gave results of sufficient acceptability for the simple needs of those early civilizations. Thus the Egyptian and Babylonian formulas for volumes of granaries and areas of land were arrived at by trial and error, with the result that many of these formulas are definitely faulty. For example, an Egyptian formula for finding the area of a circle was to take the square of eight ninths of the circle's diameter. This is not correct, as it is equivalent to taking $\pi = (4/3)^4 = 3.1604\cdots$. The even less accurate value of $\pi = 3$ is implied by some Babylonian formulas.[3] Another incorrect formula found in ancient Babylonian mathematics is one that says that the volume of a frustum of a cone or of a square pyramid is given by the product of the altitude and half the sum of the bases. It seems that the Babylonians also used, for the area of a quadrilateral having a, b, c, d for its consecutive sides, the incorrect formula $K = (a+c)(b+d)/4$. This formula gives the correct result only if the quadrilateral is a rectangle; in every other instance the formula gives too large an answer. It is curious that this same incorrect formula was reproduced 2000 years later in an Egyptian inscription found in the tomb of Ptolemy XI, who died in 51 B.C.

In general, simple empirical reasoning may be described as the formulation of conclusions based upon experience or observation; no real understanding is involved, and the logical element does not appear. Empirical reasoning often entails stodgy fiddling with special cases, observation of coincidences, experience

[2] For comments on a possible prehuman origin of mathematics see D. E. Smith [1], vol. 1, chap. 1, and H. Eves [3], Items 1°, 2°, 3°, 4°.

[3] This value for π is also found in the Bible; see I Kings 7:23, and II Chron. 4:2.

at good guessing, considerable experimentation, and flashes of intuition. Perhaps a very simple hypothetical illustration of empirical reasoning might clarify what is meant by this type of procedure.

Suppose a farmer wishes to enclose with 200 feet of fencing a rectangular field of greatest possible area along a straight river bank, no fencing being required along the river side of the field. If we designate as the *depth* of the field the dimension of the field perpendicular to the river bank and as the *length* of the field the dimension parallel to the river bank (see Figure 1.1), the farmer could soon form the following table:

Depth in feet	Length in feet	Area in square feet
10	180	1800
20	160	3200
30	140	4200
40	120	4800
50	100	5000
60	80	4800
70	60	4200
80	40	3200
90	20	1800

Examination of the table shows that the maximum area recorded occurs when the depth is 50 feet and the length is 100 feet. The interested farmer might now try various depths close to, but on each side of, 50 and would perhaps make the following additional table:

Depth	Length	Area
48	104	4992
49	102	4998
50	100	5000
51	98	4998
52	96	4992

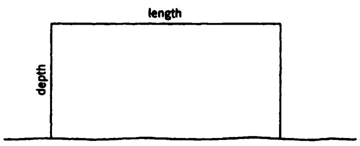

FIGURE 1.1

By now the farmer would feel quite certain that the maximum area is obtained when the depth is 50 feet and the length is 100 feet; that is, he would accept the proposition that *the maximum area occurs when the length of the field is twice the depth of the field*. A further strengthening of this belief would result from his examination of the symmetry observed in his table, and he would no doubt use his conjecture and pass it along to others as a reliable mathematical fact. Of course, the farmer's conclusion is by no means established, and no present-day student of mathematics would be permitted to "prove" the conjecture in this fashion. Shrewd guessing has taken the place of deductive logic; patience has replaced brilliance.

In spite of the empirical nature of ancient oriental mathematics, with its complete neglect of proof and the seemingly little attention paid to the difference between exact and approximate truth, one is nevertheless struck by the extent and the diversity of the problems successfully attacked. Particularly has this become evident in recent years with the scholarly deciphering of many Babylonian mathematical tablets. Apparently a great deal of elementary mathematical truth can be discovered by empirical methods when supplemented by extensive experimentation carried on patiently over a long period of time.

How were the mathematical findings of the ancient orient stated? Here we must rely on such primary sources as the Rhind, the Moscow, and other Egyptian mathematical papyri and on the approximately three hundred Babylonian mathematical tablets that have so far been deciphered.

The Rhind, or Ahmes, papyrus is a mathematical text dating from about 1650 B.C. Partaking of the nature of a practical handbook, it contains 85 problems copied by the scribe Ahmes from a still earlier work. Now possessed by the British Museum, it was originally purchased in Egypt by the Scottish antiquarian, A. Henry Rhind. This papyrus and the somewhat older Moscow papyrus, a similar mathematical text containing 25 problems, constitute our chief sources of information concerning ancient Egyptian mathematics. All of the 110 problems found in these papyri are numerical, and many of them are very easy. In general, each problem is first formulated and then followed by a step-by-step solution using the special numbers given at the beginning. Although special numbers are employed in this fashion, one feels that they are incidental and are being used merely to illustrate a general procedure. Many of the problems require nothing more than a simple linear equation, and are generally solved by the method known later in Europe as the *rule of false position*. This rule clearly reflects the empirical nature of the mathematical procedures of the time. As an example, suppose we are to solve the simple equation $x + (x/5) = 24$. Assume any convenient value for x, say, $x = 5$. Then $x + (x/5) = 6$, instead of 24. Since 6 must be multiplied by 4 to give the required 24, the correct value of x must be 4(5), or 20.

The Babylonian mathematical tablets are of two types, *table texts* and *problem texts*. There must be at least 500,000 Babylonian tablets now scattered among various museums of the world; of these only about 100 problem texts, and somewhat more than twice this number of table texts, are known to us. The table texts exhibit a wide variety of mathematical tables, such as multiplication tables, tables of reciprocals (for reducing division to multiplication), tables of squares and square roots and cubes and cube roots, tables of sums of squares and cubes (for solving certain types of cubic equations), exponential tables (for computing compound interest), and many others. The ancient Babylonians were indefatigable table makers, as one might have expected, for the construction of tables is indispensable to empirical procedure.

The problem texts also show considerable variety and are all more or less concerned with the formulation and solution of algebraic and geometric problems. A large group of the problem texts, like the Egyptian papyri considered above, formulate a problem in terms of specific numbers and then proceed with a step-by-step solution using the specific numbers. Such texts often terminate with the phrase, "such is the procedure." Again it is apparent that it is the general procedure, and not the numerical result, that is considered important. If, in a multiplication, a factor has the value 1, multiplication by this 1 will be explicitly performed, for this step is necessary in the general case. The remaining problem texts contain on a single tablet, often not as large as a page of this book, a large number of related numerical problems carefully arranged from the simplest cases up through the more complicated ones. The apparent purpose of such a text was to teach, by repetition and gradual introduction of complexities, a certain method or procedure, and the accompanying numbers serve merely as a guide to illustrate the underlying general procedure. The solution of quadratic equations, for example, both by general formula and by the method of completing the square, is explained in this way on ancient Babylonian tablets.

In summary, then, we find that pre-Hellenic mathematics was empirical. Nowhere do we find in ancient oriental mathematics a single instance of what we today call a logical demonstration. Instead of an argument we find a description of a process explained by means of specific numerical cases. In short, we are instructed to "Do thus and so." It is very interesting to note that although today confirmed students of the scientific method find this "Do thus and so" procedure highly unsatisfactory it is the procedure employed in much of our elementary teaching.

1.2 Induction Versus Deduction

Empirical conclusions, we have seen, are generalizations based on a limited number of observations or experiments. For example, the farmer of the previous section obtained a general rule by observing a limited number of computed areas. Another farmer may observe that unusually good crops have followed a number of winters of heavy snow, and empirically conclude that snowy winters are beneficial to crops. As a further example, a scientist may observe that particularly fine displays of the aurora borealis always occurred in his experience during periods of pronounced sun-spot activity and conclude that there must be a connection between the two phenomena. This type of reasoning, which concludes on the basis of a limited number of instances that something is always true, is known as *induction*.[4] Modern probability considerations have served to introduce refinements into inductive procedures. It is important to note,

[4] *Induction* should not be confused with so-called *mathematical induction*, which is considered in Section 7.3.

however, that no matter how fully the conclusions of inductive reasoning may seem warranted by the facts, these conclusions are not established beyond all possible doubt; conclusions obtained by induction are only more or less probable.

Empirical conclusions are sometimes reached by using a primitive form of induction known as reasoning by analogy. For example, if we cut off the top of a triangle by a line parallel to the base of the triangle, a trapezoid will remain, and the area of a trapezoid is given by the product of its altitude and the arithmetic average of its two bases. Now, if we cut off the top of a pyramid by a plane parallel to the base of the pyramid, a frustum will remain. By analogy one might expect the volume of a frustum of a pyramid to be given, as before, by the product of the altitude and the arithmetic average of the two bases. This is the incorrect Babylonian formula noted in the previous section. Reasoning by analogy certainly is useful, but obviously its conclusions cannot be regarded as established.

In sharp contrast to reasoning by analogy or by induction is reasoning by deduction, because the conclusions reached by deduction, provided one accepts the premises that are adopted and the system of logic that is employed, are incontestable. To illustrate deductive procedure, consider the following two statements: (1) All Canadians are North Americans; (2) Two particular men under consideration are Canadians. If we accept these two statements, we are logically compelled, following accepted principles of Aristotelian logic, to accept a third statement—namely, (3) The two men under consideration are North Americans. This is an example of deductive reasoning, which at this point may be described as those ways of deriving new statements from accepted ones that *compel* us to accept the derived statements. In the example, the first two statements are called *premises*, and the third statement the *conclusion*.

It is very important to realize that in deductive reasoning we are not concerned with the *truth* of the conclusion but rather whether the conclusion does or does not follow from the premises. If the conclusion follows from the premises, we say that our reasoning is *valid*; if it does not, we say that our reasoning is *invalid*. For example, from the two

Premises: (1) All college students are clever,
(2) All freshmen are college students,

follows the

Conclusion: All freshmen are clever.

Now the last statement certainly is not regarded generally as true, but the reasoning leading to it is valid. *If both of the premises had been true, the conclusion also would have been true*; it is essential that one understand early in the treatment of this book this meaning of the deductive process.

A useful and easy way to test the validity of a piece of deductive reasoning, like either of the examples given above, is by a diagrammatic procedure ascribed to the eminent Swiss mathematician Leonhard Euler (1707–1783). Consider our last example. We may represent the class of all clever people by a planar region within a closed boundary, and we may do likewise for the class of all college students and for the class of all freshmen. But statement (1) insists that the class of all college students is contained in the class of all clever people, and statement (2) insists that the class of all freshmen is contained in the class of all college students. Thus our various classes must be represented by their corresponding regions as shown in Figure 1.2. Clearly, the requirements of our premises *forced* us to place the class of all freshmen entirely within the class of all clever people, which is exactly what our conclusion asserts. Hence, although the conclusion is undoubtedly false, the reasoning leading to it is valid. It cannot be overemphasized at this point that the expert in the use of deduction is not fundamentally concerned with *truth* but with *validity*; he merely wants to be able to assert that his conclusions are implied by the premises. It would then follow that *if* the premises should happen to be true, the conclusion *must*, of necessity, also be true.

Consider, as a second example, the following:

Premises: (1) All parallelograms are polygons.
(2) All quadrilaterals are polygons.
Conclusion: All parallelograms are quadrilaterals.

Here all three statements are true, but the reasoning is invalid, for the premises do not *force* us to place the region representing the class of all parallelograms entirely within the region representing the class of all quadrilaterals; we are able to satisfy the requirements of our premises by a diagram like that shown in Figure 1.3.

As a third example, consider the following:

Premises: (1) All parallelograms are circles.
(2) All circles are polygons.
Conclusion: All parallelograms are polygons.

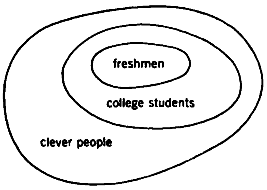

FIGURE 1.2

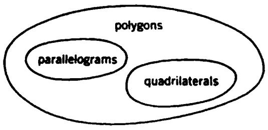

FIGURE 1.3

Here the premises are both false, the conclusion is true, and, as tested by the diagram of Figure 1.4, the reasoning is valid. Thus false assumptions may actually yield a true conclusion. True premises can yield only true conclusions when deductive logic is applied, but false premises may or may not yield true conclusions.

Finally, we shall examine the following:

Premises: (1) No quadrilaterals are triangles.
(2) Some quadrilaterals are parallelograms.
Conclusion: Some parallelograms are not triangles.

Since, by (1), the region representing the class of all quadrilaterals and that representing the class of all triangles *cannot* overlap, and, by (2), the region representing the class of all quadrilaterals and that representing the class of all parallelograms *must* overlap, the conclusion (see Figure 1.5) certainly follows, and the reasoning is valid. Note, however, that we cannot conclude, from our premises, that *no* parallelogram is a triangle, for there is nothing that *forces* us to keep the region representing the class of all parallelograms from cutting into the region representing the class of all triangles.

Euler's diagrammatic device can be used in a great variety of situations, and it is recommended to the person unfamiliar with logical procedure.

We shall not, for the present, go beyond the above superficial study of inductive and deductive reasoning. As already indicated, deductive reasoning has the advantage that its conclusions are unquestionable if the premises are accepted, and it has the additional advantage of considerable economy: before a

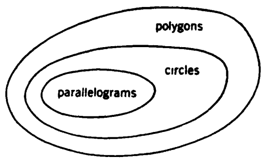

FIGURE 1.4

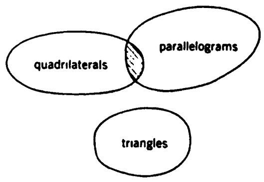

FIGURE 1.5