ELEMENTARY

LINEAR ALGEBRA

WITH APPLICATIONS

NINTH EDITION

BERNARD KOLMAN | DAVID R. HILL

Library of Congress Cataloging-in-Publication Data on file.

Editorial Director, Computer Science, Engineering, and Advanced Mathematics: *Marcia J. Horton*
Senior Editor: *Holly Stark*
Editorial Assistant: *Jennifer Lonschein*
Senior Managing Editor/Production Editor: *Scott Disanno*
Art Director: *Juan López*
Cover Designer: *Michael Fruhbeis*
Art Editor: *Thomas Benfatti*
Manufacturing Buyer: *Lisa McDowell*
Marketing Manager: *Tim Galligan*
Cover Image: (c) *William T. Williams, Artist, 1969 Trane, 1969 Acrylic on canvas, 108" x 84", Collection of The Studio Museum in Harlem. Gift of Charles Cowles, New York.*

Pearson Education, Inc.
Upper Saddle River, New Jersey 07458

Printed in the United States of America
10 9 8 7 6 5 4 3 2 1

ISBN 0-13-229654-3

Pearson Education, Ltd., *London*
Pearson Education Australia PTY. Limited, *Sydney*
Pearson Education Singapore, Pte., Ltd
Pearson Education North Asia Ltd, *Hong Kong*
Pearson Education Canada, Ltd., *Toronto*
Pearson Educación de Mexico, S.A. de C.V.
Pearson Education—Japan, *Tokyo*
Pearson Education Malaysia, Pte. Ltd

To Lisa, Stephen,
and to the memory of Lillie

B. K.

To Suzanne

D. R. H.

This page intentionally left blank

This page intentionally left blank

PREFACE

Linear algebra is an important course for a diverse number of students for at least two reasons. First, few subjects can claim to have such widespread applications in other areas of mathematics–multivariable calculus, differential equations, and probability, for example–as well as in physics, biology, chemistry, economics, finance, psychology, sociology, and all fields of engineering. Second, the subject presents the student at the sophomore level with an excellent opportunity to learn how to handle abstract concepts.

This book provides an introduction to the basic ideas and computational techniques of linear algebra at the sophomore level. It also includes a wide variety of carefully selected applications. These include topics of contemporary interest, such as Google™and Global Positioning System (GPS). The book also introduces the student to working with abstract concepts. In covering the basic ideas of linear algebra, the abstract ideas are carefully balanced by considerable emphasis on the geometrical and computational aspects of the subject. This edition continues to provide the optional opportunity to use MATLAB™or other software to enhance the pedagogy of the book.

What's New in the Ninth Edition

We have been very pleased by the wide acceptance of the first eight editions of this book throughout the 38 years of its life. In preparing this edition, we have carefully considered many suggestions from faculty and students for improving the content and presentation of the material. We have been especially gratified by hearing from the multigenerational users who used this book as students and are now using it as faculty members. Although a great many changes have been made to develop this major revision, our objective has remained the same as in the first eight editions: *to present the basic ideas of linear algebra in a manner that the student will find understandable.* To achieve this objective, the following features have been developed in this edition:

- Discussion questions have been added to the Chapter Review material. Many of these are suitable for writing projects or group activities.
- Old Section 2.1 has been split into two sections, 2.1, *Echelon Form of a Matrix*, and 2.2, *Solving Linear Systems*. This will provide improved pedagogy for covering this important material.
- Old Chapter 6, *Determinants*, has now become Chapter 3, to permit earlier coverage of this material.
- Old Section 3.4, *Span and Linear Independence*, has been split into two sections, 4.3, *Span*, and 4.4, *Linear Independence*. Since students often have difficulties with these more abstract topics, this revision presents this material at a somewhat slower pace and has more examples.

- Chapter 8, *Applications of Eigenvalues and Eigenvectors*, is new to this edition in this form. It consists of old sections 7.3, 7.6 through 7.9, material from old section 7.5 on the transmission of symmetric images, and old sections 8.1 and 8.2.
- More geometric material illustrating the discussions of diagonalization of symmetric matrices and singular value decompositions.
- Section 1.7, *Computer Graphics*, has been expanded.
- More applications have been added. These include networks and chemical balance equations.
- The exposition has been expanded in many places to improve the pedagogy and more explanations have been added to show the importance of certain material and results.
- A simplified discussion showing how linear algebra is used in global positioning systems (GPS) has been added.
- More material on recurrence relations has been added.
- More varied examples of vector spaces have been introduced.
- More material discussing the four fundamental subspaces of linear algebra have been added.
- More geometry has been added.
- More figures have been added.
- More exercises at all levels have been added.
- Exercises involving real world data have been updated to include more recent data sets.
- More MATLAB exercises have been added.

EXERCISES The exercises form an integral part of the text. Many of them are numerical in nature, whereas others are of a theoretical type. New to this edition are Discussion Exercises at the end of each of the first seven chapters, which can be used for writing projects or group activities. Many theoretical and discussion exercises, as well as some numerical ones, call for a verbal solution. In this technological age, it is especially important to be able to write with care and precision; exercises of this type should help to sharpen this skill. This edition contains almost 200 new exercises. Computer exercises, clearly indicated by a special symbol, are of two types: in the first eight chapters there are exercises allowing for discovery and exploration that do not specify any particular software to be used for their solution; in Chapter 10 there are 147 exercises designed to be solved using MATLAB. To extend the instructional capabilities of MATLAB we have developed a set of pedagogical routines, called scripts or M-files, to illustrate concepts, streamline step-by-step computational procedures, and demonstrate geometric aspects of topics using graphical displays. We feel that MATLAB and our instructional M-files provide an opportunity for a working partnership between the student and the computer that in many ways forecasts situations that will occur once a student joins the technological workforce. The exercises in this chapter are keyed to topics rather than individual sections of the text. Short descriptive headings and references to MATLAB commands in Chapter 9 supply information about the sets of exercises.

The answers to all odd-numbered exercises appear in the back of the book. An **Instructor's Solutions Manual** (ISBN: 0-13-229655-1), containing answers to all even-numbered exercises and solutions to all theoretical exercises, is available (to instructors only) from the publisher.

PRESENTATION

We have learned from experience that at the sophomore level, abstract ideas must be introduced quite gradually and must be based on firm foundations. Thus we begin the study of linear algebra with the treatment of matrices as mere arrays of numbers that arise naturally in the solution of systems of linear equations, a problem already familiar to the student. Much attention has been devoted from one edition to the next to refining and improving the pedagogical aspects of the exposition. The abstract ideas are carefully balanced by the considerable emphasis on the geometrical and computational aspects of the subject. Appendix C, *Introduction to Proofs* can be used to give the student a quick introduction to the foundations of proofs in mathematics. An expanded version of this material appears in Chapter 0 of the Student Solutions Manual.

MATERIAL COVERED

In using this book, for a one-quarter linear algebra course meeting four times a week, no difficulty has been encountered in covering eigenvalues and eigenvectors, omitting the optional material. Varying the amount of time spent on the theoretical material can readily change the level and pace of the course. Thus, the book can be used to teach a number of different types of courses.

Chapter 1 deals with matrices and their properties. In this chapter we also provide an early introduction to matrix transformations and an application of the dot product to statistics. Methods for solving systems of linear equations are discussed in **Chapter 2**. **Chapter 3** introduces the basic properties of determinants and some of their applications. In **Chapter 4**, we come to a more abstract notion, real vector spaces. Here we tap some of the many geometric ideas that arise naturally. Thus we prove that an n-dimensional, real vector space is isomorphic to R^n, the vector space of all ordered n-tuples of real numbers, or the vector space of all $n \times 1$ matrices with real entries. Since R^n is but a slight generalization of R^2 and R^3, two- and three-dimensional space are discussed at the beginning of the chapter. This shows that the notion of a finite-dimensional, real vector space is not as remote as it may have seemed when first introduced. **Chapter 5** covers inner product spaces and has a strong geometric orientation. **Chapter 6** deals with matrices and linear transformations; here we consider the dimension theorems and also applications to the solution of systems of linear equations. **Chapter 7** considers eigenvalues and eigenvectors. In this chapter we completely solve the diagonalization problem for symmetric matrices. **Chapter 8** (optional) presents an introduction to some applications of eigenvalues and eigenvectors. Section 8.3, *Dominant Eigenvalue and Principal Component Analysis*, highlights some very useful results in linear algebra. It is possible to go from Section 7.2 directly to Section 8.4, *Differential Equations*, showing how linear algebra is used to solve differential equations. Section 8.5, *Dynamical Systems* gives an application of linear algebra to an important area of modern applied mathematics. In this chapter we also discuss real quadratic forms, conic sections, and quadric surfaces. **Chapter 9**, MATLAB *for Linear Algebra*, provides an introduction to MATLAB. **Chapter 10**, MATLAB *Exercises*, consists of 147 exercises that are designed to be solved using MATLAB. **Appendix A** reviews some very basic material dealing with sets and functions. It can be consulted at any time as needed. **Appendix B**, on complex numbers, introduces in a brief but thorough manner complex numbers and their use in linear algebra. **Appendix C** provides a brief introduction to proofs in mathematics.

MATLAB SOFTWARE

The instructional M-files that have been developed to be used for solving the exercises in this book, in particular those in Chapter 9, are available on the following website: www.prenhall.com/kolman. These M-files are designed to transform many of MATLAB's capabilities into courseware. Although the computational exercises can be solved using a number of software packages, in our judgment MATLAB is the most suitable package for this purpose. MATLAB is a versatile and powerful software package whose cornerstone is its linear algebra capabilities. This is done by providing pedagogy that allows the student to interact with MATLAB, thereby letting the student think through all the steps in the solution of a problem and relegating MATLAB to act as a powerful calculator to relieve the drudgery of tedious computation. Indeed, this is the ideal role for MATLAB (or any other similar package) in a beginning linear algebra course, for in this course, more than many others, the tedium of lengthy computations makes it almost impossible to solve a modest-size problem. Thus, by introducing pedagogy and reining in the power of MATLAB, these M-files provide a working partnership between the student and the computer. Moreover, the introduction to a powerful tool such as MATLAB early in the student's college career opens the way for other software support in higher-level courses, especially in science and engineering.

MATLAB incorporates professionally developed quality computer routines for linear algebra computation. The code employed by MATLAB is written in the C language and is upgraded as new versions of MATLAB are released. MATLAB is available from The Math Works Inc., 3 Apple Hill Drive, Natick, MA 01760, e-mail: info@mathworks.com, [508-647-7000]. The Student version is available from *The Math Works* at a reasonable cost. This Student Edition of MATLAB also includes a version of Maple™, thereby providing a symbolic computational capability.

STUDENT SOLUTIONS MANUAL

The **Student Solutions Manual** (ISBN: 0-13-229656-X), prepared by Dennis R. Kletzing, Stetson University, contains solutions to all odd-numbered exercises, both numerical and theoretical.

ACKNOWLEDGMENTS

We are pleased to express our thanks to the following reviewers of the first eight editions: the late Edward Norman, University of Central Florida; the late Charles S. Duris, and Herbert J. Nichol, both at Drexel University; Stephen D. Kerr, Weber State College; Norman Lee, Ball State University; William Briggs, University of Colorado; Richard Roth, University of Colorado; David Stanford, College of William and Mary; David L. Abrahamson, Rhode Island College; Ruth Berger, Memphis State University; Michael A. Geraghty, University of Iowa; You-Feng Lin, University of South Florida; Lothar Redlin, Pennsylvania State University, Abington; Richard Sot, University of Nevada, Reno; Raymond Southworth, Professor Emeritus, College of William and Mary; J. Barry Turett, Oakland University; Gordon Brown, University of Colorado; Matt Insall, University of Missouri

at Rolla; Wolfgang Kappe State University of New York at Binghampton; Richard P. Kubelka, San Jose State University; James Nation, University of Hawaii; David Peterson, University of Central Arkansas; Malcolm J. Sherman, State University of New York at Albany, Alex Stanoyevitch, University of Hawaii; Barbara Tabak, Brandeis University; Loring W. Tu Tufts University; Manfred Kolster, McMaster University; Daniel Cunningham, Buffalo State College; Larry K. Chu, Minot State University; Daniel King, Sarah Lawrence University; Kevin Yang, Minot State University; Avy Soffer, Rutgers University; David Kaminski, University of Lethbridge, Patricia Beaulieu, University of Louisiana, Will Murray, California State University at Long Beach and of the ninth edition: Thalia D. Jeffres, Wichita State University, Manuel Lopez, Rochester Institute of Technology, Thomas L. Scofield, Calvin College, Jim Gehrmann, California State University, Sacramento, John M. Erdman, Portland State University, Ada Cheng, Kettering University, Juergen Gerlach, Radford University, and Martha Allen, Georgia College and State University.

The numerous suggestions, comments, and criticisms of these people greatly improved the manuscript.

We thank Dennis R. Kletzing, who typeset the entire manuscript, the *Student Solutions Manual*, and the *Instructor's Solutions Manual*. He found and corrected a number of mathematical errors in the manuscript. It was a pleasure working with him.

We thank Blaise deSesa for carefully checking the answers to all the exercises.

We thank William T. Williams, for kindly letting us put his striking image *Trane* on the cover of this edition.

We also thank Lilian N. Brady and Nina Edelman, Temple University, for critically and carefully reading page proofs; and instructors and students from many institutions in the United States and other countries, for sharing with us their experiences with the book for the past 38 years and offering helpful suggestions.

Finally, a sincere expression of thanks goes to Scott Disanno, Senior Managing Editor; to Holly Stark, Senior Editor; to Jennifer Lonschein, Editorial Assistant, and to the entire staff of Prentice Hall for their enthusiasm, interest, and unfailing cooperation during the conception, design, production, and marketing phases of this edition. It was a genuine pleasure working with them.

B.K.
D.R.H.

This page intentionally left blank

TO THE STUDENT

This course may be unlike any other mathematics course that you have studied thus far in at least two important ways. First, it may be your initial introduction to abstraction. Second, it is a mathematics course that may well have the greatest impact on your vocation.

Unlike other mathematics courses, this course will not give you a toolkit of isolated computational techniques for solving certain types of problems. Instead, we will develop a core of material called linear algebra by introducing certain definitions and creating procedures for determining properties and proving theorems. Proving a theorem is a skill that takes time to master, so we will develop your skill at proving mathematical results very carefully. We introduce you to abstraction slowly and amply illustrate each abstract idea with concrete numerical examples and applications. Although you will be doing a lot of computations, the goal in most problems is not merely to get the "right" answer, but to understand and be able explain how to get the answer and then interpret the result.

Linear algebra is used in the everyday world to solve problems in other areas of mathematics, physics, biology, chemistry, engineering, statistics, economics, finance, psychology, and sociology. Applications that use linear algebra include the transmission of information, the development of special effects in film and video, recording of sound, Web search engines on the Internet, global positioning system (GPS) and economic analyses. Thus, you can see how profoundly linear algebra affects you. A selected number of applications are included in this book, and if there is enough time, some of these may be covered in your course. Additionally, many of the applications can be used as self-study projects. An extensive list of applications appears in the front inside cover.

There are four different types of exercises in this book. First, there are computational exercises. These exercises and the numbers in them have been carefully chosen so that almost all of them can readily be done by hand. When you use linear algebra in real applications, you will find that the problems are much bigger in size and the numbers that occur in them are not always "nice." This is not a problem because you will almost certainly use powerful software to solve them. A taste of this type of software is provided by the third type of exercises. These are exercises designed to be solved by using a computer and MATLAB™, a powerful matrix-based application that is widely used in industry. The second type of exercises are theoretical. Some of these may ask you to prove a result or discuss an idea. The fourth type of exercises are discussion exercises, which can be used as group projects. In today's world, it is not enough to be able to compute an answer; you often have to prepare a report discussing your solution, justifying the steps in your solution, and interpreting your results. These types of exercises will give you experience in writing mathematics. Mathematics uses words, not just symbols.

How to Succeed in Linear Algebra

- Read the book slowly with pencil and paper at hand. You might have to read a particular section more than once. Take the time to verify the steps marked "verify" in the text.
- Make sure to do your homework on a timely basis. If you wait until the problems are explained in class, you will miss learning how to solve a problem by yourself. Even if you can't complete a problem, try it anyway, so that when you see it done in class you will understand it more easily. You might find it helpful to work with other students on the material covered in class and on some homework problems.
- Make sure that you ask for help as soon as something is not clear to you. Each abstract idea in this course is based on previously developed ideas—much like laying a foundation and then building a house. If any of the ideas are fuzzy to you or missing, your knowledge of the course will not be sturdy enough for you to grasp succeeding ideas.
- Make use of the pedagogical tools provided in this book. At the end of each section in the first eight chapters, we have a list of key terms; at the end of each of the first seven chapters we have a chapter review, supplementary exercises,a chapter quiz, and discussion exercises. Answers to the odd-numbered computational exercises appear at the end of the book. The Student Solutions Manual provides detailed solutions to all odd-numbered exercises, both numerical and theoretical. It can be purchased from the publisher (ISBN 0-13-229656-X).

We assure you that your efforts to learn linear algebra well will be amply rewarded in other courses and in your professional career.

We wish you much success in your study of linear algebra.

Bernard Kolman

David R. Hill

CHAPTER

1 Linear Equations and Matrices

1.1 Systems of Linear Equations

One of the most frequently recurring practical problems in many fields of study—such as mathematics, physics, biology, chemistry, economics, all phases of engineering, operations research, and the social sciences—is that of solving a system of linear equations. The equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b, \tag{1}$$

which expresses the real or complex quantity b in terms of the unknowns x_1, x_2, $\ldots$, x_n and the real or complex constants $a_1, a_2, \ldots, a_n$, is called a **linear equation.** In many applications we are given b and must find numbers $x_1, x_2, \ldots, x_n$ satisfying (1).

A **solution** to linear Equation (1) is a sequence of n numbers $s_1, s_2, \ldots, s_n$, which has the property that (1) is satisfied when $x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$ are substituted in (1). Thus $x_1 = 2$, $x_2 = 3$, and $x_3 = -4$ is a solution to the linear equation

$$6x_1 - 3x_2 + 4x_3 = -13,$$

because

$$6(2) - 3(3) + 4(-4) = -13.$$

More generally, a **system of m linear equations in n unknowns**, $x_1, x_2, \ldots, x_n$, or a **linear system**, is a set of m linear equations each in n unknowns. A linear

Note: Appendix A reviews some very basic material dealing with sets and functions. It can be consulted at any time, as needed.

system can conveniently be written as

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad\quad\; \vdots \quad &\;\; \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m.
\end{aligned} \tag{2}$$

Thus the ith equation is

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i.$$

In (2) the a_{ij} are known constants. Given values of $b_1, b_2, \ldots, b_m$, we want to find values of $x_1, x_2, \ldots, x_n$ that will satisfy each equation in (2).

A **solution** to linear system (2) is a sequence of n numbers $s_1, s_2, \ldots, s_n$, which has the property that each equation in (2) is satisfied when $x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$ are substituted.

If the linear system (2) has no solution, it is said to be **inconsistent**; if it has a solution, it is called **consistent**. If $b_1 = b_2 = \cdots = b_m = 0$, then (2) is called a **homogeneous system**. Note that $x_1 = x_2 = \cdots = x_n = 0$ is always a solution to a homogeneous system; it is called the **trivial solution**. A solution to a homogeneous system in which not all of $x_1, x_2, \ldots, x_n$ are zero is called a **nontrivial solution**.

Consider another system of r linear equations in n unknowns:

$$\begin{aligned}
c_{11}x_1 + c_{12}x_2 + \cdots + c_{1n}x_n &= d_1 \\
c_{21}x_1 + c_{22}x_2 + \cdots + c_{2n}x_n &= d_2 \\
\vdots \qquad \vdots \qquad\quad\; \vdots \quad &\;\; \vdots \\
c_{r1}x_1 + c_{r2}x_2 + \cdots + c_{rn}x_n &= d_r.
\end{aligned} \tag{3}$$

We say that (2) and (3) are **equivalent** if they both have exactly the same solutions.

EXAMPLE 1 The linear system

$$\begin{aligned}
x_1 - 3x_2 &= -7 \\
2x_1 + x_2 &= 7
\end{aligned} \tag{4}$$

has only the solution $x_1 = 2$ and $x_2 = 3$. The linear system

$$\begin{aligned}
8x_1 - 3x_2 &= 7 \\
3x_1 - 2x_2 &= 0 \\
10x_1 - 2x_2 &= 14
\end{aligned} \tag{5}$$

also has only the solution $x_1 = 2$ and $x_2 = 3$. Thus (4) and (5) are equivalent. ■

To find a solution to a linear system, we shall use a technique called the **method of elimination**; that is, we eliminate some variables by adding a multiple of one equation to another equation. Elimination merely amounts to the development of a new linear system that is equivalent to the original system, but is much simpler to solve. Readers have probably confined their earlier work in this area to linear systems in which $m = n$, that is, linear systems having as many equations as unknowns. In this course we shall broaden our outlook by dealing with systems in which we have $m = n$, $m < n$, and $m > n$. Indeed, there are numerous applications in which $m \neq n$. If we deal with two, three, or four unknowns, we shall often write them as x, y, z, and w. In this section we use the method of elimination as it was studied in high school. In Section 2.2 we shall look at this method in a much more systematic manner.

EXAMPLE 2

The director of a trust fund has \$100,000 to invest. The rules of the trust state that both a certificate of deposit (CD) and a long-term bond must be used. The director's goal is to have the trust yield \$7800 on its investments for the year. The CD chosen returns 5% per annum, and the bond 9%. The director determines the amount x to invest in the CD and the amount y to invest in the bond as follows:

Since the total investment is \$100,000, we must have $x + y = 100{,}000$. Since the desired return is \$7800, we obtain the equation $0.05x + 0.09y = 7800$. Thus, we have the linear system

$$\begin{aligned} x + y &= 100{,}000 \\ 0.05x + 0.09y &= 7800. \end{aligned} \tag{6}$$

To eliminate x, we add (-0.05) times the first equation to the second, obtaining

$$0.04y = 2800,$$

an equation having no x term. We have eliminated the unknown x. Then solving for y, we have

$$y = 70{,}000,$$

and substituting into the first equation of (6), we obtain

$$x = 30{,}000.$$

To check that $x = 30{,}000$, $y = 70{,}000$ is a solution to (6), we verify that these values of x and y satisfy *each* of the equations in the given linear system. This solution is the only solution to (6); the system is consistent. The director of the trust should invest \$30,000 in the CD and \$70,000 in the long-term bond. ■

EXAMPLE 3

Consider the linear system

$$\begin{aligned} x - 3y &= -7 \\ 2x - 6y &= 7. \end{aligned} \tag{7}$$

Again, we decide to eliminate x. We add (-2) times the first equation to the second one, obtaining

$$0 = 21,$$

which makes no sense. This means that (7) has no solution; it is inconsistent. We could have come to the same conclusion from observing that in (7) the left side of the second equation is twice the left side of the first equation, but the right side of the second equation is not twice the right side of the first equation. ■

EXAMPLE 4 Consider the linear system

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2. \end{aligned} \tag{8}$$

To eliminate x, we add (-2) times the first equation to the second one and (-3) times the first equation to the third one, obtaining

$$\begin{aligned} -7y - 4z &= 2 \\ -5y - 10z &= -20. \end{aligned} \tag{9}$$

This is a system of two equations in the unknowns y and z. We multiply the second equation of (9) by $\left(-\frac{1}{5}\right)$, yielding

$$\begin{aligned} -7y - 4z &= 2 \\ y + 2z &= 4, \end{aligned}$$

which we write, by interchanging equations, as

$$\begin{aligned} y + 2z &= 4 \\ -7y - 4z &= 2. \end{aligned} \tag{10}$$

We now eliminate y in (10) by adding 7 times the first equation to the second one, to obtain

$$10z = 30,$$

or

$$z = 3. \tag{11}$$

Substituting this value of z into the first equation of (10), we find that $y = -2$. Then substituting these values of y and z into the first equation of (8), we find that $x = 1$. We observe further that our elimination procedure has actually produced the linear system

$$\begin{aligned} x + 2y + 3z &= 6 \\ y + 2z &= 4 \\ z &= 3, \end{aligned} \tag{12}$$

obtained by using the first equations of (8) and (10) as well as (11). The importance of this procedure is that, although the linear systems (8) and (12) are equivalent, (12) has the advantage that it is easier to solve. ■

EXAMPLE 5 Consider the linear system

$$\begin{aligned} x + 2y - 3z &= -4 \\ 2x + y - 3z &= 4. \end{aligned} \tag{13}$$

Eliminating x, we add (-2) times the first equation to the second equation to get

$$-3y + 3z = 12. \tag{14}$$

We must now solve (14). A solution is

$$y = z - 4,$$

where z can be any real number. Then from the first equation of (13),

$$\begin{aligned} x &= -4 - 2y + 3z \\ &= -4 - 2(z - 4) + 3z \\ &= z + 4. \end{aligned}$$

Thus a solution to the linear system (13) is

$$\begin{aligned} x &= z + 4 \\ y &= z - 4 \\ z &= \text{any real number.} \end{aligned}$$

This means that the linear system (13) has infinitely many solutions. Every time we assign a value to z we obtain another solution to (13). Thus, if $z = 1$, then

$$x = 5, \quad y = -3, \quad \text{and} \quad z = 1$$

is a solution, while if $z = -2$, then

$$x = 2, \quad y = -6, \quad \text{and} \quad z = -2$$

is another solution. ■

These examples suggest that a linear system may have a unique solution, no solution, or infinitely many solutions.

Consider next a linear system of two equations in the unknowns x and y:

$$\begin{aligned} a_1x + a_2y &= c_1 \\ b_1x + b_2y &= c_2. \end{aligned} \tag{15}$$

The graph of each of these equations is a straight line, which we denote by ℓ_1 and ℓ_2, respectively. If $x = s_1$, $y = s_2$ is a solution to the linear system (15), then the point (s_1, s_2) lies on both lines ℓ_1 and ℓ_2. Conversely, if the point (s_1, s_2) lies on both lines ℓ_1 and ℓ_2, then $x = s_1$, $y = s_2$ is a solution to the linear system (15). Thus we are led geometrically to the same three possibilities mentioned previously. See Figure 1.1.

Next, consider a linear system of three equations in the unknowns x, y, and z:

$$\begin{aligned} a_1x - b_1y + c_1z &= d_1 \\ a_2x - b_2y + c_2z &= d_2 \\ a_3x - b_3y + c_3z &= d_3. \end{aligned} \tag{16}$$

The graph of each of these equations is a plane, denoted by P_1, P_2, and P_3, respectively. As in the case of a linear system of two equations in two unknowns,

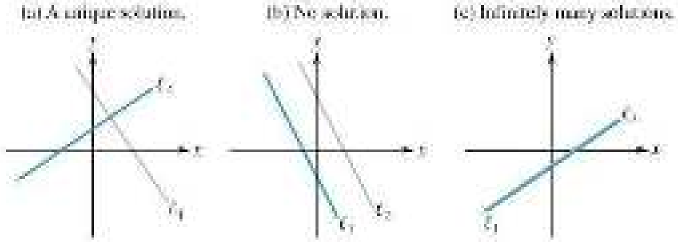

FIGURE 1.1

the linear system in (16) can have infinitely many solutions, a unique solution, or no solution. These situations are illustrated in Figure 1.2. For a more concrete illustration of some of the possible cases, consider that two intersecting walls and the ceiling (planes) of a room intersect in a unique point, a corner of the room, so the linear system has a unique solution. Next, think of the planes as pages of a book. Three pages of a book (held open) intersect in a straight line, the spine. Thus, the linear system has infinitely many solutions. On the other hand, when the book is closed, three pages of a book appear to be parallel and do not intersect, so the linear system has no solution.

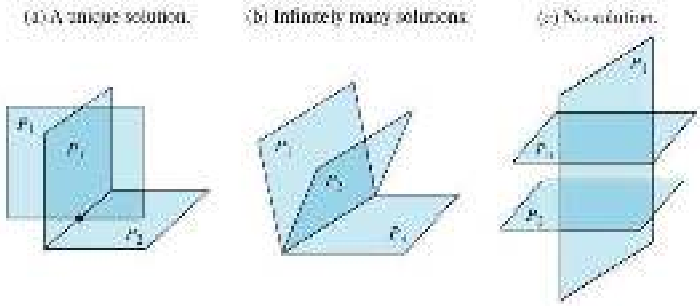

FIGURE 1.2

If we examine the method of elimination more closely, we find that it involves three manipulations that can be performed on a linear system to convert it into an equivalent system. These manipulations are as follows:

1. Interchange the ith and jth equations.
2. Multiply an equation by a nonzero constant.
3. Replace the ith equation by c times the jth equation plus the ith equation, $i \neq j$. That is, replace

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i$$

by

$$(ca_{j1} + a_{i1})x_1 + (ca_{j2} + a_{i2})x_2 + \cdots + (ca_{jn} + a_{in})x_n = cb_j + b_i$$

It is not difficult to prove that performing these manipulations on a linear system leads to an equivalent system. The next example proves this for the second type of manipulation. Exercises 24 and 25 prove it for the first and third manipulations, respectively.

EXAMPLE 6

Suppose that the ith equation of the linear system (2) is multiplied by the nonzero constant c, producing the linear system

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ ca_{i1}x_1 & + & ca_{i2}x_2 & + & \cdots & + & ca_{in}x_n & = & cb_i \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m. \end{array} \tag{17}$$

If $x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$ is a solution to (2), then it is a solution to all the equations in (17), except possibly to the ith equation. For the ith equation we have

$$c(a_{i1}s_1 + a_{i2}s_2 + \cdots + a_{in}s_n) = cb_i$$

or

$$ca_{i1}s_1 + ca_{i2}s_2 + \cdots + ca_{in}s_n = cb_i.$$

Thus the ith equation of (17) is also satisfied. Hence every solution to (2) is also a solution to (17). Conversely, every solution to (17) also satisfies (2). Hence (2) and (17) are equivalent systems. ■

The following example gives an application leading to a linear system of two equations in three unknowns:

EXAMPLE 7

(Production Planning) A manufacturer makes three different types of chemical products: A, B, and C. Each product must go through two processing machines: X and Y. The products require the following times in machines X and Y:

1. One ton of A requires 2 hours in machine X and 2 hours in machine Y.
2. One ton of B requires 3 hours in machine X and 2 hours in machine Y.
3. One ton of C requires 4 hours in machine X and 3 hours in machine Y.

Machine X is available 80 hours per week, and machine Y is available 60 hours per week. Since management does not want to keep the expensive machines X and Y idle, it would like to know how many tons of each product to make so that the machines are fully utilized. It is assumed that the manufacturer can sell as much of the products as is made.

To solve this problem, we let x_1, x_2, and x_3 denote the number of tons of products A, B, and C, respectively, to be made. The number of hours that machine X will be used is

$$2x_1 + 3x_2 + 4x_3,$$

which must equal 80. Thus we have

$$2x_1 + 3x_2 + 4x_3 = 80.$$

Similarly, the number of hours that machine Y will be used is 60, so we have

$$2x_1 + 2x_2 + 3x_3 = 60.$$

Mathematically, our problem is to find nonnegative values of x_1, x_2, and x_3 so that

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &= 80 \\ 2x_1 + 2x_2 + 3x_3 &= 60. \end{aligned}$$

This linear system has infinitely many solutions. Following the method of Example 4, we see that all solutions are given by

$$\begin{aligned} x_1 &= \frac{20 - x_3}{2} \\ x_2 &= 20 - x_3 \\ x_3 &= \text{any real number such that } 0 \le x_3 \le 20, \end{aligned}$$

since we must have $x_1 \ge 0$, $x_2 \ge 0$, and $x_3 \ge 0$. When $x_3 = 10$, we have

$$x_1 = 5, \qquad x_2 = 10, \qquad x_3 = 10$$

while

$$x_1 = \tfrac{13}{2}, \qquad x_2 = 13, \qquad x_3 = 7$$

when $x_3 = 7$. The reader should observe that one solution is just as good as the other. There is no best solution unless additional information or restrictions are given. ■

As you have probably already observed, the method of elimination has been described, so far, in general terms. Thus we have not indicated any rules for selecting the unknowns to be eliminated. Before providing a very systematic description of the method of elimination, we introduce in the next section the notion of a matrix. This will greatly simplify our notational problems and will enable us to develop tools to solve many important applied problems.

Key Terms

Linear equation	Consistent system	Unique solution
Solution of a linear equation	Homogeneous system	No solution
Linear system	Trivial solution	Infinitely many solutions
Unknowns	Nontrivial solution	Manipulations on linear systems
Inconsistent system	Equivalent systems	Method of elimination

1.1 Exercises

In Exercises 1 through 14, solve each given linear system by the method of elimination.

1. $\begin{aligned} x + 2y &= 8 \\ 3x - 4y &= 4 \end{aligned}$

2. $\begin{aligned} 2x - 3y + 4z &= -12 \\ x - 2y + z &= -5 \\ 3x + y + 2z &= 1 \end{aligned}$

3. $\begin{aligned} 3x + 2y + z &= 2 \\ 4x + 2y + 2z &= 8 \\ x - y + z &= 4 \end{aligned}$

4. $\begin{aligned} x + y &= 5 \\ 3x + 3y &= 10 \end{aligned}$

5. $\begin{aligned} 2x + 4y + 6z &= -12 \\ 2x - 3y - 4z &= 15 \\ 3x + 4y + 5z &= -8 \end{aligned}$

6. $\begin{aligned} x + y - 2z &= 5 \\ 2x + 3y + 4z &= 2 \end{aligned}$

7. $\begin{aligned} x + 4y - z &= 12 \\ 3x + 8y - 2z &= 4 \end{aligned}$

8. $\begin{aligned} 3x + 4y - z &= 8 \\ 6x + 8y - 2z &= 3 \end{aligned}$

9. $\begin{aligned} x + y + 3z &= 12 \\ 2x + 2y + 6z &= 6 \end{aligned}$

10. $\begin{aligned} x + y &= 1 \\ 2x - y &= 5 \\ 3x + 4y &= 2 \end{aligned}$

11. $\begin{aligned} 2x + 3y &= 13 \\ x - 2y &= 3 \\ 5x + 2y &= 27 \end{aligned}$

12. $\begin{aligned} x - 5y &= 6 \\ 3x + 2y &= 1 \\ 5x + 2y &= 1 \end{aligned}$

13. $\begin{aligned} x + 3y &= -4 \\ 2x + 5y &= -8 \\ x + 3y &= -5 \end{aligned}$

14. $\begin{aligned} 2x + 3y - z &= 6 \\ 2x - y + 2z &= -8 \\ 3x - y + z &= -7 \end{aligned}$

15. Given the linear system

$$\begin{aligned} 2x - y &= 5 \\ 4x - 2y &= t, \end{aligned}$$

(a) Determine a particular value of t so that the system is consistent.

(b) Determine a particular value of t so that the system is inconsistent.

(c) How many different values of t can be selected in part (b)?

16. Given the linear system

$$\begin{aligned} 3x + 4y &= s \\ 6x + 8y &= t, \end{aligned}$$

(a) Determine particular values for s and t so that the system is consistent.

(b) Determine particular values for s and t so that the system is inconsistent.

(c) What relationship between the values of s and t will guarantee that the system is consistent?

17. Given the linear system

$$\begin{aligned} x - 2y &= 10 \\ 3x - (6 + t)y &= 30, \end{aligned}$$

(a) Determine a particular value of t so that the system has infinitely many solutions.

(b) Determine a particular value of t so that the system has a unique solution.

(c) How many different values of t can be selected in part (b)?

18. Is every homogeneous linear system always consistent? Explain.

19. Given the linear system

$$\begin{aligned} 2x + 3y - z &= 0 \\ x - 4y + 5z &= 0, \end{aligned}$$

(a) Verify that $x_1 = 1$, $y_1 = -1$, $z_1 = -1$ is a solution.

(b) Verify that $x_2 = -2$, $y_2 = 2$, $z_2 = 2$ is a solution.

(c) Is $x = x_1 + x_2 = -1$, $y = y_1 + y_2 = 1$, and $z = z_1 + z_2 = 1$ a solution to the linear system?

(d) Is $3x$, $3y$, $3z$, where x, y, and z are as in part (c), a solution to the linear system?

20. Without using the method of elimination, solve the linear system

$$\begin{aligned} 2x + y - 2z &= -5 \\ 3y + z &= 7 \\ z &= 4. \end{aligned}$$

21. Without using the method of elimination, solve the linear system

$$\begin{aligned} 4x &= 8 \\ -2x + 3y &= -1 \\ 3x + 5y - 2z &= 11. \end{aligned}$$

22. Is there a value of r so that $x = 1$, $y = 2$, $z = r$ is a solution to the following linear system? If there is, find it.

$$\begin{aligned} 2x + 3y - z &= 11 \\ x - y + 2z &= -7 \\ 4x + y - 2z &= 12. \end{aligned}$$

23. Is there a value of r so that $x = r$, $y = 2$, $z = 1$ is a solution to the following linear system? If there is, find it.

$$\begin{aligned} 3x - 2z &= 4 \\ x - 4y + z &= -5 \\ -2x + 3y + 2z &= 9. \end{aligned}$$

24. Show that the linear system obtained by interchanging two equations in (2) is equivalent to (2).

25. Show that the linear system obtained by adding a multiple of an equation in (2) to another equation is equivalent to (2).

26. Describe the number of points that simultaneously lie in each of the three planes shown in each part of Figure 1.2.

27. Describe the number of points that simultaneously lie in each of the three planes shown in each part of Figure 1.3.

28. Let C_1 and C_2 be circles in the plane. Describe the number of possible points of intersection of C_1 and C_2. Illustrate each case with a figure.

29. Let S_1 and S_2 be spheres in space. Describe the number of possible points of intersection of S_1 and S_2. Illustrate each case with a figure.

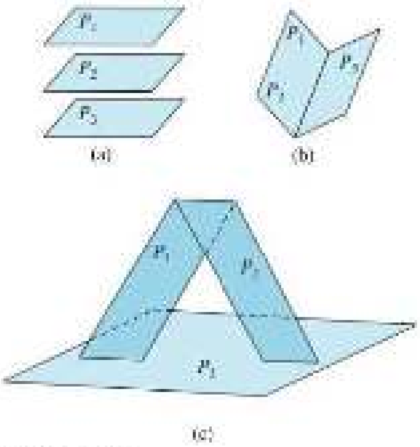

FIGURE 1.3

30. An oil refinery produces low-sulfur and high-sulfur fuel. Each ton of low-sulfur fuel requires 5 minutes in the blending plant and 4 minutes in the refining plant; each ton of high-sulfur fuel requires 4 minutes in the blending plant and 2 minutes in the refining plant. If the blending plant is available for 3 hours and the refining plant is available for 2 hours, how many tons of each type of fuel should be manufactured so that the plants are fully used?

31. A plastics manufacturer makes two types of plastic: regular and special. Each ton of regular plastic requires 2 hours in plant A and 5 hours in plant B; each ton of special plastic requires 2 hours in plant A and 3 hours in plant B. If plant A is available 8 hours per day and plant B is available 15 hours per day, how many tons of each type of plastic can be made daily so that the plants are fully used?

32. A dietician is preparing a meal consisting of foods A, B, and C. Each ounce of food A contains 2 units of protein, 3 units of fat, and 4 units of carbohydrate. Each ounce of food B contains 3 units of protein, 2 units of fat, and 1 unit of carbohydrate. Each ounce of food C contains 3 units of protein, 3 units of fat, and 2 units of carbohydrate. If the meal must provide exactly 25 units of protein, 24 units of fat, and 21 units of carbohydrate, how many ounces of each type of food should be used?

33. A manufacturer makes 2-minute, 6-minute, and 9-minute film developers. Each ton of 2-minute developer requires 6 minutes in plant A and 24 minutes in plant B. Each ton of 6-minute developer requires 12 minutes in plant A and 12 minutes in plant B. Each ton of 9-minute developer requires 12 minutes in plant A and 12 minutes in plant B. If plant A is available 10 hours per day and plant B is available 16 hours per day, how many tons of each type of developer can be produced so that the plants are fully used?

34. Suppose that the three points $(1, -5)$, $(-1, 1)$, and $(2, 7)$ lie on the parabola $p(x) = ax^2 + bx + c$.
   (a) Determine a linear system of three equations in three unknowns that must be solved to find a, b, and c.
   (b) Solve the linear system obtained in part (a) for a, b, and c.

35. An inheritance of \$24,000 is to be divided among three trusts, with the second trust receiving twice as much as the first trust. The three trusts pay interest annually at the rates of 9%, 10%, and 6%, respectively, and return a total in interest of \$2210 at the end of the first year. How much was invested in each trust?

36. For the software you are using, determine the command that "automatically" solves a linear system of equations.

37. Use the command from Exercise 36 to solve Exercises 3 and 4, and compare the output with the results you obtained by the method of elimination.

38. Solve the linear system

$$\begin{aligned} x + \tfrac{1}{2}y + \tfrac{1}{3}z &= 1 \\ \tfrac{1}{2}x + \tfrac{1}{3}y + \tfrac{1}{4}z &= \tfrac{11}{18} \\ \tfrac{1}{3}x + \tfrac{1}{4}y + \tfrac{1}{5}z &= \tfrac{9}{20} \end{aligned}$$

by using your software. Compare the computed solution with the exact solution $x = \frac{1}{2}$, $y = \frac{1}{3}$, $z = 1$.

39. If your software includes access to a computer algebra system (CAS), use it as follows:
   (a) For the linear system in Exercise 38, replace the fraction $\frac{1}{2}$ with its decimal equivalent 0.5. Enter this system into your software and use the appropriate CAS commands to solve the system. Compare the solution with that obtained in Exercise 38.
   (b) In some CAS environments you can select the number of digits to be used in the calculations. Perform part (a) with digit choices 2, 4, and 6 to see what influence such selections have on the computed solution.

40. If your software includes access to a CAS and you can select the number of digits used in calculations, do the following: Enter the linear system

$$\begin{aligned} 0.71x + 0.21y &= 0.92 \\ 0.23x + 0.58y &= 0.81 \end{aligned}$$

into the program. Have the software solve the system with digit choices 2, 5, 7, and 12. Briefly discuss any variations in the solutions generated.

1.2 Matrices

If we examine the method of elimination described in Section 1.1, we can make the following observation: Only the numbers in front of the unknowns $x_1, x_2, \ldots, x_n$ and the numbers $b_1, b_2, \ldots, b_m$ on the right side are being changed as we perform the steps in the method of elimination. Thus we might think of looking for a way of writing a linear system without having to carry along the unknowns. Matrices enable us to do this—that is, to write linear systems in a compact form that makes it easier to automate the elimination method by using computer software in order to obtain a fast and efficient procedure for finding solutions. The use of matrices, however, is not merely that of a convenient notation. We now develop operations on matrices and will work with matrices according to the rules they obey; this will enable us to solve systems of linear equations and to handle other computational problems in a fast and efficient manner. Of course, as any good definition should do, the notion of a matrix not only provides a new way of looking at old problems, but also gives rise to a great many new questions, some of which we study in this book.

DEFINITION 1.1 An $m \times n$ **matrix** A is a rectangular array of mn real or complex numbers arranged in m horizontal **rows** and n vertical **columns**:

$$A = \begin{array}{c} \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & a_{ij} & \vdots \\ \vdots & \vdots & \cdots & \cdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & \cdots & \cdots & a_{mn} \end{bmatrix} \begin{array}{l} \\ \\ \\ \leftarrow i\text{th row} \\ \\ \\ \end{array} \\ \uparrow j\text{th column} \end{array} \tag{1}$$

The **ith row** of A is

$$\begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \qquad (1 \le i \le m);$$

the **jth column** of A is

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} \qquad (1 \le j \le n).$$

We shall say that A is **m by n** (written as $m \times n$). If $m = n$, we say that A is a **square matrix of order n**, and that the numbers $a_{11}, a_{22}, \ldots, a_{nn}$ form the **main diagonal** of A. We refer to the number a_{ij}, which is in the ith row and jth column of A, as the **i,jth element** of A, or the **(i, j) entry** of A, and we often write (1) as

$$A = \begin{bmatrix} a_{ij} \end{bmatrix}.$$