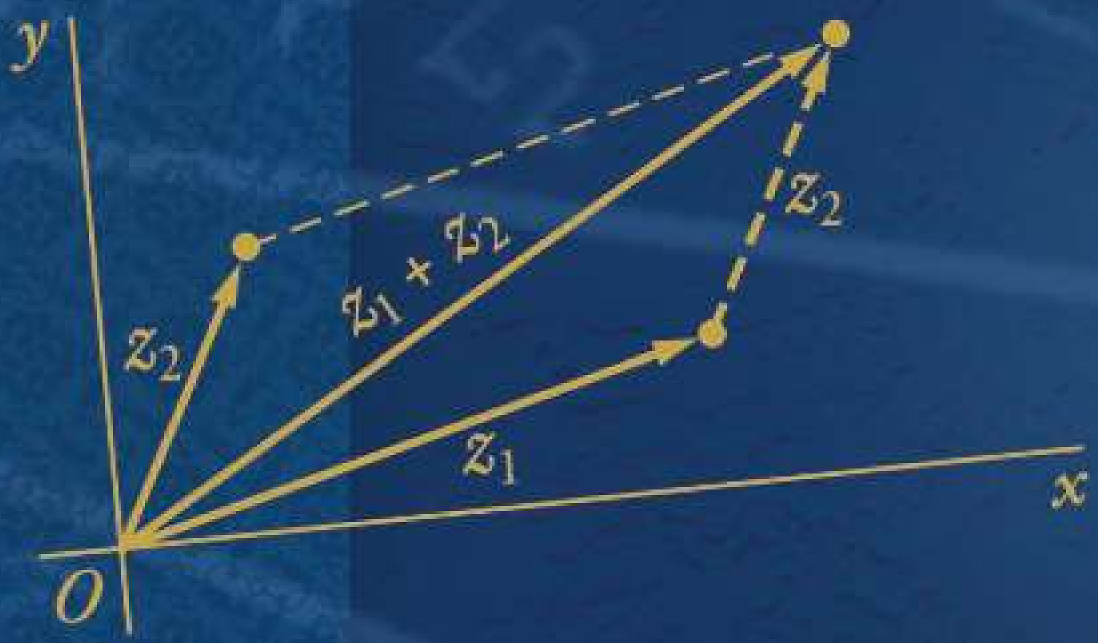
Eighth Edition

Complex Variables
and Applications



James Ward Brown
Ruel V. Churchill

$$|z_1 + z_2| \le |z_1| + |z_2|$$

COMPLEX VARIABLES AND APPLICATIONS

Eighth Edition

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McGraw-Hill Higher Education

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ABOUT THE AUTHORS

JAMES WARD BROWN is Professor of Mathematics at The University of Michigan– Dearborn. He earned his A.B. in physics from Harvard University and his A.M. and Ph.D. in mathematics from The University of Michigan in Ann Arbor, where he was an Institute of Science and Technology Predoctoral Fellow. He is coauthor with Dr. Churchill of *Fourier Series and Boundary Value Problems*, now in its seventh edition. He has received a research grant from the National Science Foundation as well as a Distinguished Faculty Award from the Michigan Association of Governing Boards of Colleges and Universities. Dr. Brown is listed in *Who's Who in the World*.

RUEL V. CHURCHILL was, at the time of his death in 1987, Professor Emeritus of Mathematics at The University of Michigan, where he began teaching in 1922. He received his B.S. in physics from the University of Chicago and his M.S. in physics and Ph.D. in mathematics from The University of Michigan. He was coauthor with Dr. Brown of *Fourier Series and Boundary Value Problems*, a classic text that he first wrote almost 70 years ago. He was also the author of *Operational Mathematics*. Dr. Churchill held various offices in the Mathematical Association of America and in other mathematical societies and councils.

To the Memory of My Father
George H. Brown

and of My Long-Time Friend and Coauthor
Ruel V. Churchill

These Distinguished Men of Science for Years Influenced
The Careers of Many People, Including Myself.

JWB

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PREFACE

This book is a revision of the seventh edition, which was published in 2004. That edition has served, just as the earlier ones did, as a textbook for a one-term introductory course in the theory and application of functions of a complex variable. This new edition preserves the basic content and style of the earlier editions, the first two of which were written by the late Ruel V. Churchill alone.

The *first objective* of the book is to develop those parts of the theory that are prominent in applications of the subject. The *second objective* is to furnish an introduction to applications of residues and conformal mapping. With regard to residues, special emphasis is given to their use in evaluating real improper integrals, finding inverse Laplace transforms, and locating zeros of functions. As for conformal mapping, considerable attention is paid to its use in solving boundary value problems that arise in studies of heat conduction and fluid flow. Hence the book may be considered as a companion volume to the authors' text "Fourier Series and Boundary Value Problems," where another classical method for solving boundary value problems in partial differential equations is developed.

The first nine chapters of this book have for many years formed the basis of a three-hour course given each term at The University of Michigan. The classes have consisted mainly of seniors and graduate students concentrating in mathematics, engineering, or one of the physical sciences. Before taking the course, the students have completed at least a three-term calculus sequence and a first course in ordinary differential equations. Much of the material in the book need not be covered in the lectures and can be left for self-study or used for reference. If mapping by elementary functions is desired earlier in the course, one can skip to Chap. 8 immediately after Chap. 3 on elementary functions.

In order to accommodate as wide a range of readers as possible, there are footnotes referring to other texts that give proofs and discussions of the more delicate results from calculus and advanced calculus that are occasionally needed. A bibliography of other books on complex variables, many of which are more advanced, is provided in Appendix 1. A table of conformal transformations that are useful in applications appears in Appendix 2.

The main changes in this edition appear in the first nine chapters. Many of those changes have been suggested by users of the last edition. Some readers have urged that sections which can be skipped or postponed without disruption be more clearly identified. The statements of Taylor's theorem and Laurent's theorem, for example, now appear in sections that are separate from the sections containing their proofs. Another significant change involves the extended form of the Cauchy integral formula for derivatives. The treatment of that extension has been completely rewritten, and its immediate consequences are now more focused and appear together in a single section.

Other improvements that seemed necessary include more details in arguments involving mathematical induction, a greater emphasis on rules for using complex exponents, some discussion of residues at infinity, and a clearer exposition of real improper integrals and their Cauchy principal values. In addition, some rearrangement of material was called for. For instance, the discussion of upper bounds of moduli of integrals is now entirely in one section, and there is a separate section devoted to the definition and illustration of isolated singular points. Exercise sets occur more frequently than in earlier editions and, as a result, concentrate more directly on the material at hand.

Finally, there is an *Student's Solutions Manual* (ISBN: 978-0-07-333730-2; MHID: 0-07-333730-7) that is available upon request to instructors who adopt the book. It contains solutions of selected exercises in Chapters 1 through 7, covering the material through residues.

In the preparation of this edition, continual interest and support has been provided by a variety of people, especially the staff at McGraw-Hill and my wife Jacqueline Read Brown.

James Ward Brown

CHAPTER

1

COMPLEX NUMBERS

In this chapter, we survey the algebraic and geometric structure of the complex number system. We assume various corresponding properties of real numbers to be known.

1. SUMS AND PRODUCTS

Complex numbers can be defined as ordered pairs (x, y) of real numbers that are to be interpreted as points in the *complex plane*, with rectangular coordinates x and y, just as real numbers x are thought of as points on the real line. When real numbers x are displayed as points $(x, 0)$ on the *real axis*, it is clear that the set of complex numbers includes the real numbers as a subset. Complex numbers of the form $(0, y)$ correspond to points on the y axis and are called *pure imaginary numbers* when $y \neq 0$. The y axis is then referred to as the *imaginary axis*.

It is customary to denote a complex number (x, y) by z, so that (see Fig. 1)

(1) $$z = (x, y).$$

The real numbers x and y are, moreover, known as the *real and imaginary parts* of z, respectively; and we write

(2) $$x = \operatorname{Re} z, \; y = \operatorname{Im} z.$$

Two complex numbers z_1 and z_2 are *equal* whenever they have the same real parts and the same imaginary parts. Thus the statement $z_1 = z_2$ means that z_1 and z_2 correspond to the same point in the complex, or z, plane.

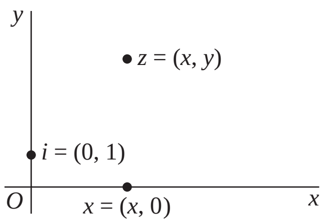

FIGURE 1

The *sum* $z_1 + z_2$ and *product* $z_1 z_2$ of two complex numbers

$$z_1 = (x_1, y_1) \quad \text{and} \quad z_2 = (x_2, y_2)$$

are defined as follows:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2), \tag{3}$$

$$(x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, y_1 x_2 + x_1 y_2). \tag{4}$$

Note that the operations defined by equations (3) and (4) become the usual operations of addition and multiplication when restricted to the real numbers:

$$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0),$$

$$(x_1, 0)(x_2, 0) = (x_1 x_2, 0).$$

The complex number system is, therefore, a natural extension of the real number system.

Any complex number $z = (x, y)$ can be written $z = (x, 0) + (0, y)$, and it is easy to see that $(0, 1)(y, 0) = (0, y)$. Hence

$$z = (x, 0) + (0, 1)(y, 0);$$

and if we think of a real number as either x or $(x, 0)$ and let i denote the pure imaginary number (0,1), as shown in Fig. 1, it is clear that*

$$z = x + iy. \tag{5}$$

Also, with the convention that $z^2 = zz$, $z^3 = z^2 z$, etc., we have

$$i^2 = (0, 1)(0, 1) = (-1, 0),$$

or

$$i^2 = -1. \tag{6}$$

*In electrical engineering, the letter j is used instead of i.

Because $(x, y) = x + iy$, definitions (3) and (4) become

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2), \tag{7}$$

$$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2). \tag{8}$$

Observe that the right-hand sides of these equations can be obtained by formally manipulating the terms on the left as if they involved only real numbers and by replacing i^2 by -1 when it occurs. Also, observe how equation (8) tells us that *any complex number times zero is zero*. More precisely,

$$z \cdot 0 = (x + iy)(0 + i0) = 0 + i0 = 0$$

for any $z = x + iy$.

2. BASIC ALGEBRAIC PROPERTIES

Various properties of addition and multiplication of complex numbers are the same as for real numbers. We list here the more basic of these algebraic properties and verify some of them. Most of the others are verified in the exercises.

The commutative laws

$$z_1 + z_2 = z_2 + z_1, \quad z_1z_2 = z_2z_1 \tag{1}$$

and the associative laws

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3), \quad (z_1z_2)z_3 = z_1(z_2z_3) \tag{2}$$

follow easily from the definitions in Sec. 1 of addition and multiplication of complex numbers and the fact that real numbers obey these laws. For example, if

$$z_1 = (x_1, y_1) \quad \text{and} \quad z_2 = (x_2, y_2),$$

then

$$z_1 + z_2 = (x_1 + x_2, y_1 + y_2) = (x_2 + x_1, y_2 + y_1) = z_2 + z_1.$$

Verification of the rest of the above laws, as well as the distributive law

$$z(z_1 + z_2) = zz_1 + zz_2, \tag{3}$$

is similar.

According to the commutative law for multiplication, $iy = yi$. Hence one can write $z = x + yi$ instead of $z = x + iy$. Also, because of the associative laws, a sum $z_1 + z_2 + z_3$ or a product $z_1z_2z_3$ is well defined without parentheses, as is the case with real numbers.

The additive identity $0 = (0, 0)$ and the multiplicative identity $1 = (1, 0)$ for real numbers carry over to the entire complex number system. That is,

$$z + 0 = z \quad \text{and} \quad z \cdot 1 = z \tag{4}$$

for every complex number z. Furthermore, 0 and 1 are the only complex numbers with such properties (see Exercise 8).

There is associated with each complex number $z = (x, y)$ an additive inverse

$$-z = (-x, -y), \tag{5}$$

satisfying the equation $z + (-z) = 0$. Moreover, there is only one additive inverse for any given z, since the equation

$$(x, y) + (u, v) = (0, 0)$$

implies that

$$u = -x \quad \text{and} \quad v = -y.$$

For any *nonzero* complex number $z = (x, y)$, there is a number z^{-1} such that $zz^{-1} = 1$. This multiplicative inverse is less obvious than the additive one. To find it, we seek real numbers u and v, expressed in terms of x and y, such that

$$(x, y)(u, v) = (1, 0).$$

According to equation (4), Sec. 1, which defines the product of two complex numbers, u and v must satisfy the pair

$$xu - yv = 1, \quad yu + xv = 0$$

of linear simultaneous equations; and simple computation yields the unique solution

$$u = \frac{x}{x^2 + y^2}, \quad v = \frac{-y}{x^2 + y^2}.$$

So *the* multiplicative inverse of $z = (x, y)$ is

$$z^{-1} = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2}\right) \qquad (z \neq 0). \tag{6}$$

The inverse z^{-1} is not defined when $z = 0$. In fact, $z = 0$ means that $x^2 + y^2 = 0$; and this is not permitted in expression (6).

in Sec. 2. Inasmuch as such properties continue to be anticipated because they also apply to real numbers, the reader can easily pass to Sec. 4 without serious disruption.

We begin with the observation that the existence of multiplicative inverses enables us to show that *if a product* z_1z_2 *is zero, then so is at least one of the factors* z_1 *and* z_2. For suppose that $z_1z_2 = 0$ and $z_1 \neq 0$. The inverse z_1^{-1} exists; and any complex number times zero is zero (Sec. 1). Hence

$$z_2 = z_2 \cdot 1 = z_2(z_1z_1^{-1}) = (z_1^{-1}z_1)z_2 = z_1^{-1}(z_1z_2) = z_1^{-1} \cdot 0 = 0.$$

That is, if $z_1z_2 = 0$, either $z_1 = 0$ or $z_2 = 0$; or possibly both of the numbers z_1 and z_2 are zero. Another way to state this result is that *if two complex numbers* z_1 *and* z_2 *are nonzero, then so is their product* z_1z_2.

Subtraction and division are defined in terms of additive and multiplicative inverses:

(1) $$z_1 - z_2 = z_1 + (-z_2),$$

(2) $$\frac{z_1}{z_2} = z_1z_2^{-1} \qquad (z_2 \neq 0).$$

Thus, in view of expressions (5) and (6) in Sec. 2,

(3) $$z_1 - z_2 = (x_1, y_1) + (-x_2, -y_2) = (x_1 - x_2, y_1 - y_2)$$

and

(4) $$\frac{z_1}{z_2} = (x_1, y_1)\left(\frac{x_2}{x_2^2 + y_2^2}, \frac{-y_2}{x_2^2 + y_2^2}\right) = \left(\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}, \frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}\right)$$
$$(z_2 \neq 0)$$

when $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$.

Using $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, one can write expressions (3) and (4) here as

(5) $$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$$

and

(6) $$\frac{z_1}{z_2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\,\frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2} \qquad (z_2 \neq 0).$$

Although expression (6) is not easy to remember, it can be obtained by writing (see Exercise 7)

(7) $$\frac{z_1}{z_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)},$$

EXERCISES

1. Verify that

(a) $(\sqrt{2} - i) - i(1 - \sqrt{2}i) = -2i$; *(b)* $(2, -3)(-2, 1) = (-1, 8)$;

(c) $(3, 1)(3, -1)\left(\frac{1}{5}, \frac{1}{10}\right) = (2, 1)$.

2. Show that

(a) $\operatorname{Re}(iz) = -\operatorname{Im} z$; *(b)* $\operatorname{Im}(iz) = \operatorname{Re} z$.

3. Show that $(1 + z)^2 = 1 + 2z + z^2$.

4. Verify that each of the two numbers $z = 1 \pm i$ satisfies the equation $z^2 - 2z + 2 = 0$.

5. Prove that multiplication of complex numbers is commutative, as stated at the beginning of Sec. 2.

6. Verify

(a) the associative law for addition of complex numbers, stated at the beginning of Sec. 2;

(b) the distributive law (3), Sec. 2.

7. Use the associative law for addition and the distributive law to show that

$$z(z_1 + z_2 + z_3) = zz_1 + zz_2 + zz_3.$$

8. *(a)* Write $(x, y) + (u, v) = (x, y)$ and point out how it follows that the complex number $0 = (0, 0)$ is unique as an additive identity.

(b) Likewise, write $(x, y)(u, v) = (x, y)$ and show that the number $1 = (1, 0)$ is a unique multiplicative identity.

9. Use $-1 = (-1, 0)$ and $z = (x, y)$ to show that $(-1)z = -z$.

10. Use $i = (0, 1)$ and $y = (y, 0)$ to verify that $-(iy) = (-i)y$. Thus show that the additive inverse of a complex number $z = x + iy$ can be written $-z = -x - iy$ without ambiguity.

11. Solve the equation $z^2 + z + 1 = 0$ for $z = (x, y)$ by writing

$$(x, y)(x, y) + (x, y) + (1, 0) = (0, 0)$$

and then solving a pair of simultaneous equations in x and y.

Suggestion: Use the fact that no real number x satisfies the given equation to show that $y \neq 0$.

Ans. $z = \left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$.

3. FURTHER PROPERTIES

In this section, we mention a number of other algebraic properties of addition and multiplication of complex numbers that follow from the ones already described multiplying out the products in the numerator and denominator on the right, and then using the property

$$\frac{z_1 + z_2}{z_3} = (z_1 + z_2)z_3^{-1} = z_1 z_3^{-1} + z_2 z_3^{-1} = \frac{z_1}{z_3} + \frac{z_2}{z_3} \qquad (z_3 \neq 0). \tag{8}$$

The motivation for starting with equation (7) appears in Sec. 5.

EXAMPLE. The method is illustrated below:

$$\frac{4+i}{2-3i} = \frac{(4+i)(2+3i)}{(2-3i)(2+3i)} = \frac{5+14i}{13} = \frac{5}{13} + \frac{14}{13}i.$$

There are some expected properties involving quotients that follow from the relation

$$\frac{1}{z_2} = z_2^{-1} \qquad (z_2 \neq 0), \tag{9}$$

which is equation (2) when $z_1 = 1$. Relation (9) enables us, for instance, to write equation (2) in the form

$$\frac{z_1}{z_2} = z_1\left(\frac{1}{z_2}\right) \qquad (z_2 \neq 0). \tag{10}$$

Also, by observing that (see Exercise 3)

$$(z_1 z_2)(z_1^{-1} z_2^{-1}) = (z_1 z_1^{-1})(z_2 z_2^{-1}) = 1 \qquad (z_1 \neq 0, z_2 \neq 0),$$

and hence that $z_1^{-1} z_2^{-1} = (z_1 z_2)^{-1}$, one can use relation (9) to show that

$$\left(\frac{1}{z_1}\right)\left(\frac{1}{z_2}\right) = z_1^{-1} z_2^{-1} = (z_1 z_2)^{-1} = \frac{1}{z_1 z_2} \qquad (z_1 \neq 0, z_2 \neq 0). \tag{11}$$

Another useful property, to be derived in the exercises, is

$$\left(\frac{z_1}{z_3}\right)\left(\frac{z_2}{z_4}\right) = \frac{z_1 z_2}{z_3 z_4} \qquad (z_3 \neq 0, z_4 \neq 0). \tag{12}$$

Finally, we note that the *binomial formula* involving real numbers remains valid with complex numbers. That is, if z_1 and z_2 are any two nonzero complex numbers, then

$$(z_1 + z_2)^n = \sum_{k=0}^{n} \binom{n}{k} z_1^k z_2^{n-k} \qquad (n = 1, 2, \ldots) \tag{13}$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad (k = 0, 1, 2, \ldots, n)$$

and where it is agreed that $0! = 1$. The proof is left as an exercise.

EXERCISES

1. Reduce each of these quantities to a real number:

$$(a)\ \frac{1+2i}{3-4i}+\frac{2-i}{5i}; \qquad (b)\ \frac{5i}{(1-i)(2-i)(3-i)}; \qquad (c)\ (1-i)^4.$$

Ans. *(a)* $-2/5$; *(b)* $-1/2$; *(c)* -4.

2. Show that

$$\frac{1}{1/z}=z \qquad (z\neq 0).$$

3. Use the associative and commutative laws for multiplication to show that

$$(z_1z_2)(z_3z_4)=(z_1z_3)(z_2z_4).$$

4. Prove that if $z_1z_2z_3=0$, then at least one of the three factors is zero.

Suggestion: Write $(z_1z_2)z_3=0$ and use a similar result (Sec. 3) involving two factors.

5. Derive expression (6), Sec. 3, for the quotient z_1/z_2 by the method described just after it.

6. With the aid of relations (10) and (11) in Sec. 3, derive the identity

$$\left(\frac{z_1}{z_3}\right)\left(\frac{z_2}{z_4}\right)=\frac{z_1z_2}{z_3z_4} \qquad (z_3\neq 0,\ z_4\neq 0).$$

7. Use the identity obtained in Exercise 6 to derive the cancellation law

$$\frac{z_1z}{z_2z}=\frac{z_1}{z_2} \qquad (z_2\neq 0,\ z\neq 0).$$

8. Use mathematical induction to verify the binomial formula (13) in Sec. 3. More precisely, note that the formula is true when $n=1$. Then, assuming that it is valid when $n=m$ where m denotes any positive integer, show that it must hold when $n=m+1$.

Suggestion: When $n=m+1$, write

$$\begin{aligned}(z_1+z_2)^{m+1}&=(z_1+z_2)(z_1+z_2)^m=(z_2+z_1)\sum_{k=0}^{m}\binom{m}{k}z_1^kz_2^{m-k}\\&=\sum_{k=0}^{m}\binom{m}{k}z_1^kz_2^{m+1-k}+\sum_{k=0}^{m}\binom{m}{k}z_1^{k+1}z_2^{m-k}\end{aligned}$$

and replace k by $k-1$ in the last sum here to obtain

$$(z_1+z_2)^{m+1}=z_2^{m+1}+\sum_{k=1}^{m}\left[\binom{m}{k}+\binom{m}{k-1}\right]z_1^kz_2^{m+1-k}+z_1^{m+1}.$$

Finally, show how the right-hand side here becomes

$$z_2^{m+1} + \sum_{k=1}^{m} \binom{m+1}{k} z_1^k z_2^{m+1-k} + z_1^{m+1} = \sum_{k=0}^{m+1} \binom{m+1}{k} z_1^k z_2^{m+1-k}.$$

4. VECTORS AND MODULI

It is natural to associate any nonzero complex number $z = x + iy$ with the directed line segment, or vector, from the origin to the point (x, y) that represents z in the complex plane. In fact, we often refer to z as the point z or the vector z. In Fig. 2 the numbers $z = x + iy$ and $-2 + i$ are displayed graphically as both points and radius vectors.

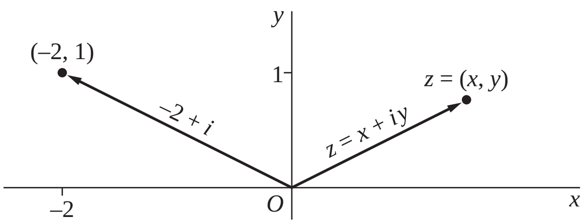

FIGURE 2

When $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, the sum

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

corresponds to the point $(x_1 + x_2, y_1 + y_2)$. It also corresponds to a vector with those coordinates as its components. Hence $z_1 + z_2$ may be obtained vectorially as shown in Fig. 3.

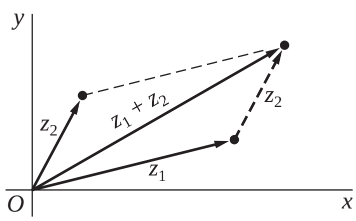

FIGURE 3

Although the product of two complex numbers z_1 and z_2 is itself a complex number represented by a vector, that vector lies in the same plane as the vectors for z_1 and z_2. Evidently, then, this product is neither the scalar nor the vector product used in ordinary vector analysis.

The vector interpretation of complex numbers is especially helpful in extending the concept of absolute values of real numbers to the complex plane. The *modulus*, or absolute value, of a complex number $z = x + iy$ is defined as the nonnegative real number $\sqrt{x^2 + y^2}$ and is denoted by $|z|$; that is,

$$|z| = \sqrt{x^2 + y^2}. \tag{1}$$

Geometrically, the number $|z|$ is the distance between the point (x, y) and the origin, or the length of the radius vector representing z. It reduces to the usual absolute value in the real number system when $y = 0$. Note that while *the inequality* $z_1 < z_2$ *is meaningless unless both* z_1 *and* z_2 *are real*, the statement $|z_1| < |z_2|$ means that the point z_1 is closer to the origin than the point z_2 is.

EXAMPLE 1. Since $|-3 + 2i| = \sqrt{13}$ and $|1 + 4i| = \sqrt{17}$, we know that the point $-3 + 2i$ is closer to the origin than $1 + 4i$ is.

The distance between two points (x_1, y_1) and (x_2, y_2) is $|z_1 - z_2|$. This is clear from Fig. 4, since $|z_1 - z_2|$ is the length of the vector representing the number

$$z_1 - z_2 = z_1 + (-z_2);$$

and, by translating the radius vector $z_1 - z_2$, one can interpret $z_1 - z_2$ as the directed line segment from the point (x_2, y_2) to the point (x_1, y_1). Alternatively, it follows from the expression

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$$

and definition (1) that

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

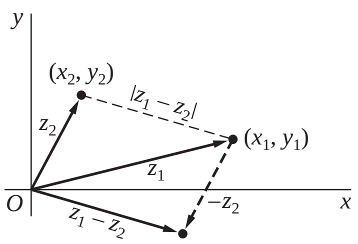

FIGURE 4

The complex numbers z corresponding to the points lying on the circle with center z_0 and radius R thus satisfy the equation $|z - z_0| = R$, and conversely. We refer to this set of points simply as the circle $|z - z_0| = R$.

EXAMPLE 2. The equation $|z - 1 + 3i| = 2$ represents the circle whose center is $z_0 = (1, -3)$ and whose radius is $R = 2$.

It also follows from definition (1) that the real numbers $|z|$, $\operatorname{Re} z = x$, and $\operatorname{Im} z = y$ are related by the equation

$$|z|^2 = (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2. \tag{2}$$

Thus

$$\operatorname{Re} z \le |\operatorname{Re} z| \le |z| \quad \text{and} \quad \operatorname{Im} z \le |\operatorname{Im} z| \le |z|. \tag{3}$$

We turn now to the *triangle inequality*, which provides an upper bound for the modulus of the sum of two complex numbers z_1 and z_2:

$$|z_1 + z_2| \le |z_1| + |z_2|. \tag{4}$$

This important inequality is geometrically evident in Fig. 3, since it is merely a statement that the length of one side of a triangle is less than or equal to the sum of the lengths of the other two sides. We can also see from Fig. 3 that inequality (4) is actually an equality when 0, z_1, and z_2 are collinear. Another, strictly algebraic, derivation is given in Exercise 15, Sec. 5.

An immediate consequence of the triangle inequality is the fact that

$$|z_1 + z_2| \ge ||z_1| - |z_2||. \tag{5}$$

To derive inequality (5), we write

$$|z_1| = |(z_1 + z_2) + (-z_2)| \le |z_1 + z_2| + |-z_2|,$$

which means that

$$|z_1 + z_2| \ge |z_1| - |z_2|. \tag{6}$$

This is inequality (5) when $|z_1| \ge |z_2|$. If $|z_1| < |z_2|$, we need only interchange z_1 and z_2 in inequality (6) to arrive at

$$|z_1 + z_2| \ge -(|z_1| - |z_2|),$$

which is the desired result. Inequality (5) tells us, of course, that the length of one side of a triangle is greater than or equal to the difference of the lengths of the other two sides.

Because $|-z_2| = |z_2|$, one can replace z_2 by $-z_2$ in inequalities (4) and (5) to summarize these results in a particularly useful form:

$$|z_1 \pm z_2| \le |z_1| + |z_2|, \tag{7}$$

$$|z_1 \pm z_2| \ge ||z_1| - |z_2||. \tag{8}$$

When combined, inequalities (7) and (8) become

$$||z_1| - |z_2|| \le |z_1 \pm z_2| \le |z_1| + |z_2|. \tag{9}$$