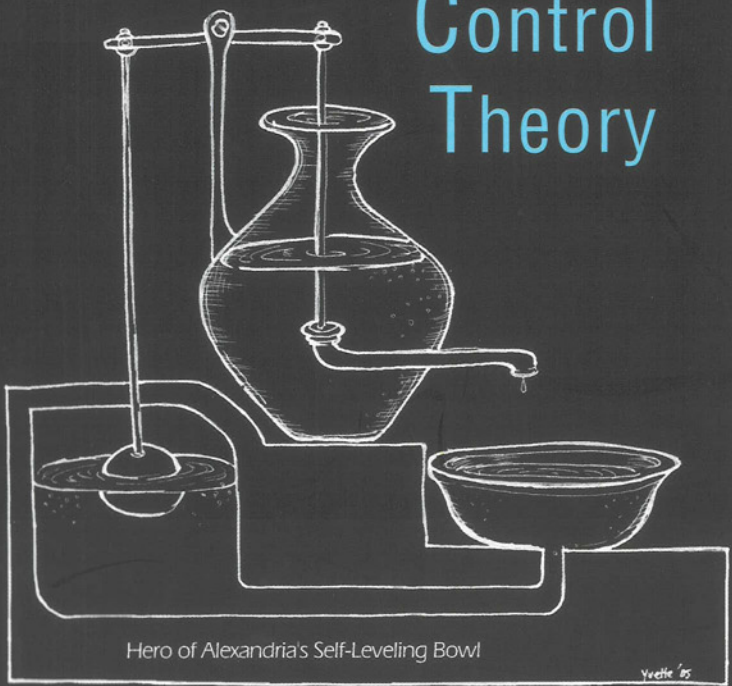
Shlomo Engelberg

VOL. 2

A Mathematical Introduction to Control Theory



Hero of Alexandria's Self-Leveling Bowl

SERIES IN ELECTRICAL AND COMPUTER ENGINEERING

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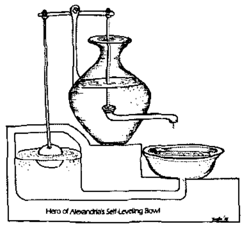

A Mathematical Introduction to Control Theory

SERIES IN ELECTRICAL AND COMPUTER ENGINEERING

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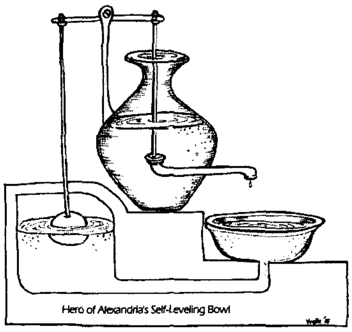

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Dedication

This book is dedicated to the memory of my beloved uncle
Stephen Aaron Engelberg (1940-2005)
who helped teach me how a *mensch* behaves
and how a person can love and appreciate learning.
May his memory be a blessing.

Preface

Control theory is largely an application of the theory of complex variables, modern algebra, and linear algebra to engineering. The main question that control theory answers is "given reasonable inputs, will my system give reasonable outputs?" Much of the answer to this question is given in the following pages. There are many books that cover control theory. What distinguishes this book is that it provides a complete introduction to control theory without sacrificing either the intuitive side of the subject or mathematical rigor. This book shows how control theory fits into the worlds of mathematics and engineering.

This book was written for students who have had at least one semester of complex analysis and some acquaintance with ordinary differential equations. Theorems from modern algebra are quoted before use—a course in modern algebra *is not* a prerequisite for this book; a single course in complex analysis is. Additionally, to properly understand the material on modern control a first course in linear algebra is necessary. Finally, sections 5.3 and 6.4 are a bit technical in nature; they can be skipped without affecting the flow of the chapters in which they are contained.

In order to make this book as accessible as possible many footnotes have been added in places where the reader's background—either in mathematics or in engineering—may not be sufficient to understand some concept or follow some chain of reasoning. The footnotes generally add some detail that is not directly related to the argument being made. Additionally, there are several footnotes that give biographical information about the people whose names appear in these pages—often as part of the name of some technique. We hope that these footnotes will give the reader something of a feel for the history of control theory.

In the first seven chapters of this book classical control theory is developed. The next three chapters constitute an introduction to three important areas of control theory: nonlinear control, modern control, and the control of hybrid systems. The final chapter contains solutions to some of the exercises. The first seven chapters can be covered in a reasonably paced one semester course. To cover the whole book will probably take most students and instructors two semesters.

The first chapter of this book is an introduction to the Laplace transform, a brief introduction to the notion of stability, and a short introduction to MATLAB. MATLAB is used throughout this book as a very fancy calculator. MATLAB allows students to avoid some of the work that would once have had to be done by hand but which cannot be done by a person with either the speed or the accuracy with which a computer can do the same work.

The second chapter bridges the gap between the world of mathematics and of engineering. In it we present transfer functions, and we discuss how to use and manipulate block diagrams. The discussion is in sufficient depth for the non-engineer, and is hopefully not too long for the engineering student who may have been exposed to some of the material previously.

Next we introduce feedback systems. We describe how one calculates the transfer function of a feedback system. We provide a number of examples of how the overall transfer function of a system is calculated. We also discuss the sensitivity of feedback systems to their components. We discuss the conditions under which feedback control systems track their input. Finally we consider the effect of the feedback connection on the way the system deals with noise.

The next chapter is devoted to the Routh-Hurwitz Criterion. We state and prove the Routh-Hurwitz theorem—a theorem which gives a necessary and sufficient condition for the zeros of a real polynomial to be in the left half plane. We provide a number of applications of the theorem to the design of control systems.

In the fifth chapter, we cover the principle of the argument and its consequences. We start the chapter by discussing and proving the principle of the argument. We show how it leads to a graphical method—the Nyquist plot—for determining the stability of a system. We discuss low-pass systems, and we introduce the Bode plots and show how one can use them to determine the stability of such systems. We discuss the gain and phase margins and some of their limitations.

In the sixth chapter, we discuss the root locus diagram. Having covered a large portion of the classical frequency domain techniques for analyzing and designing feedback systems, we turn our attention to time-domain based approaches. We describe how one plots a root locus diagram. We explain the mathematics behind this plot—how the properties of the plot are simply properties of quotients of polynomials with real coefficients. We explain how one uses a root locus plot to analyze and design feedback systems.

In the seventh chapter we describe how one designs compensators for linear systems. Having devoted five chapters largely to the analysis of systems, in this chapter we concentrate on how to design systems. We discuss how one can use various types of compensators to improve the performance of a given system. In particular, we discuss phase-lag, phase-lead, lag-lead and PID (position integral derivative) controllers and how to use them.

In the eighth chapter we discuss nonlinear systems, limit cycles, the describing function technique, and Tsypkin's method. We show how the describing function is a very natural, albeit not always a very good, way of analyzing nonlinear circuits. We describe how one uses it to predict the existence and stability of limit cycles. We point out some of the limitations of the technique. Then we present Tsypkin's method which is an exact method but which is only useful for predicting the existence of limit cycles in a rather limited class of systems.

In the ninth chapter we consider modern control theory. We review the necessary background from linear algebra, and we carefully explain controllability and observability. Then we give necessary and sufficient conditions for controllability and observability of single-input single-output system. We also discuss the pole placement problem.

In the tenth chapter we consider discrete-time control theory and the control of hybrid systems. We start with the necessary background about the z-transform. Then we show how to analyze discrete-time system. The role of the unit circle is described, and the bilinear transform is carefully explained. We describe how to design compensators for discrete-time systems, and we give a brief introduction to the modified z-transform.

In the final chapter we provide solutions to selected exercises. The solutions are generally done at sufficient length that the student will not have to struggle too much to understand them. It is hoped that these solutions will be used instead of going to a friend or teacher to check one's answer. They should not be used to avoid thinking about how to go about solving the exercise or to avoid the real work of calculating the solution. In order to develop a good grasp of control theory, one must do problems. It is not enough to "understand" the material that has been presented; one must *experience* it.

Having spent many years preparing this book and having been helped by many people with this book, I have many people to thank. I am particularly grateful to Professors Richard G. Costello, Jonathan Goodman, Steven Schochet, and Aryeh Weiss who each read this work, critiqued it, and helped me improve it. I also grateful to the many anonymous referees whose comments helped me to improve my presentation of the beautiful results herein described.

I am happy to acknowledge Professor George Anastassiou's support. Professor Anastassiou has both encouraged me in my efforts to have this work published and has helped me in my search for a suitable publisher. My officemate, Aharon Naiman, has earned my thanks many, many times; he has helped me become more proficient in my use of LaTeX, put up with my enthusiasms, and helped me clarify my thoughts on many points.

My wife, Yvette, and my children, Chananel, Nediva, and Oriya, have always been supportive of my efforts; without Yvette's support this book would not have been written. My students been kind enough to put up with my penchant for handing out notes in English without complaining too bitterly; their comments have helped improve this book in many ways. My parents have, as always, been pillars of support. Without my father's love and appreciation of mathematics and science and my mother's love of good writing I would neither have desired to nor been suited to write a book of this nature. Because of the support of my parents, wife, children, colleagues, and students, writing this book has been a pleasant and meaningful as well as an interesting and challenging experience.

Though all of the many people who have helped and supported me over the years have made their mark on this work I, stubborn as ever, made the final decisions as to what material to include and how to present that material. The nicely turned phrase may well have been provided by a friend or mentor, by a parent or colleague; the mistakes are my own.

Shlomo Engelberg
Jerusalem, Israel

Contents

Chapter 1

Mathematical Preliminaries

1.1 An Introduction to the Laplace Transform

Much of this chapter is devoted to describing and deriving some of the properties of the one-sided Laplace transform. The Laplace transform is the engineer's most important tool for analyzing the stability of linear, time-invariant, continuous-time systems. The Laplace transform is defined as:

$$\mathcal{L}(f(t))(s) \equiv \int_0^\infty e^{-st} f(t)\, dt.$$

We often write $F(s)$ for the Laplace transform of $f(t)$. It is customary to use lower-case letters for functions of time, t, and to use the same letter—but in its upper-case form—for the Laplace transform of the function; throughout this book, we follow this practice.

We assume that the functions $f(t)$ are of *exponential type*—that they satisfy an inequality of the form $|f(t)| \leq Ce^{\alpha t}, C \in \mathcal{R}$. If the real part of s, $\Re(s)$, satisfies $\Re(s) < -\alpha$, then the integral that defines the Laplace transform converges. The Laplace transform's usefulness comes largely from the fact that it allows us to convert differential and integro-differential equations into algebraic equations.

We now calculate the Laplace transform of some functions. We start with the unit step function (also known as the Heaviside [1] function):

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}.$$

[1]After Oliver Heaviside (1850-1925) who between 1880 and 1887 invented the "operational calculus"[OR]. His operational calculus was widely used in its time. The Laplace transform that is used today is a "cousin" of Heaviside's operational calculus[Dea97].

From the definition of the Laplace transform, we find that:

$$\begin{aligned} U(s) &= \mathcal{L}(u(t))(s) \\ &= \int_0^\infty e^{-st} \cdot 1\, dt \\ &= \left.\frac{e^{-st}}{-s}\right|_0^\infty \\ &= \lim_{t\to\infty} \frac{e^{-st}}{-s} - \frac{1}{-s}. \end{aligned}$$

Denote the real part of s by α and its imaginary part by β. Continuing our calculation, we find that:

$$\begin{aligned} U(s) &= \lim_{t\to\infty} e^{-\alpha t}\frac{e^{-j\beta t}}{-s} + \frac{1}{s} \\ &= 0 + \frac{1}{s} = \frac{1}{s}. \end{aligned}$$

This holds as long as $\alpha > 0$. In this case the first term in the limit:

$$\lim_{t\to\infty} e^{-\alpha t}\frac{e^{-j\beta t}}{-s}$$

is approaching zero while the second term—though oscillatory—is bounded. In general, we assume that s is chosen so that integrals and limits that must converge do. For our purposes, the region of convergence (in terms of s) of the integral is not terribly important.

Next we consider $\mathcal{L}(e^{at})(s)$. We find that:

$$\begin{aligned} \mathcal{L}(e^{at})(s) &= \int_0^\infty e^{-st}e^{at}\, dt \\ &= \left.\frac{e^{(a-s)t}}{a-s}\right|_0^\infty \\ &= \frac{1}{s-a}. \end{aligned}$$

1.2 Properties of the Laplace Transform

The first property of the Laplace transform is its *linearity.*

Theorem 1

$$\mathcal{L}\left(\alpha f(t) + \beta g(t)\right)(s) = \alpha F(s) + \beta G(s).$$

Simply put, "the Laplace transform of a linear combination is the linear combination of the Laplace transforms."

PROOF: Making use of the properties of the integral, we find that:

$$\begin{aligned}\mathcal{L}\left(\alpha f(t) + \beta g(t)\right)(s) &= \int_0^\infty e^{-st}\left(\alpha f(t) + \beta g(t)\right)\,dt \\ &= \alpha \int_0^\infty e^{-st} f(t)\,dt + \beta \int_0^\infty e^{-st} g(t)\,dt \\ &= \alpha F(s) + \beta G(s).\end{aligned}$$

We see that the linearity of the Laplace transform is part of its "inheritance" from the integral which defines it.

The Laplace Transform of $\sin(t)$ I—An Example

Following the engineering convention that $j \equiv \sqrt{-1}$, we write:

$$\sin(t) = \frac{e^{jt} - e^{-jt}}{2j}.$$

By linearity we find that:

$$\mathcal{L}(\sin(t))(s) = \frac{1}{2j}\left(\mathcal{L}(e^{jt})(s) - \mathcal{L}(e^{-jt})(s)\right).$$

Making use of the fact that we know what the Laplace transform of an exponential is, we find that:

$$\mathcal{L}(\sin(t))(s) = \frac{1}{2j}\left(\frac{1}{s-j} - \frac{1}{s+j}\right) = \frac{1}{s^2+1}.$$

The next property we consider is the property that makes the Laplace transform transform so useful. As we shall see, it is possible to calculate the Laplace transform of the solution of a constant-coefficient ordinary differential equation (ODE) *without solving the ODE.*

Theorem 2 *Assume that $f(t)$ is has a well defined limit as t approaches zero from the right*[2]*. Then we find that:*

$$\mathcal{L}(f'(t))(s) = sF(s) - f(0^+).$$

PROOF: This result is proved by making use of integration by parts. We see that:

$$\mathcal{L}(f'(t))(s) = \int_0^\infty e^{-st} f'(t)\, dt.$$

Let $u = e^{-st}$ and $dv = f'(t)dt$. Then $du = -se^{-st}$ and $v = f(t)$. Assuming that $\alpha = \mathcal{R}(s) > 0$, we find that:

$$\begin{aligned}
\int_0^\infty e^{-st} f'(t)\, dt &= -\int_0^\infty \frac{d}{dt} e^{-st} f(t)\, dt + e^{-st} f(t)|_{0+}^{\infty} \\
&= s\int_0^\infty e^{-st} f(t)\, dt + \lim_{t\to\infty} e^{-st} f(t) - f(0^+) \\
&= sF(s) + 0 - f(0^+) \\
&= sF(s) - f(0^+).
\end{aligned}$$

We take the limit of $f(t)$ as $t \to 0^+$ because the integral itself deals only with positive values of t. Often we dispense with the added generality that the limit from the right gives us, and we write $f(0)$.

We can use this theorem to find the Laplace transform of the second (or higher) derivative of a function. To find the Laplace transform of the second derivative of a function, one applies the theorem twice. I.e.:

$$\begin{aligned}
\mathcal{L}(f''(t))(s) &= s\mathcal{L}(f'(t))(s) - f'(0) \\
&= s(sF(s) - f(0)) - f'(0) \\
&= s^2 F(s) - sf'(0) - f(0).
\end{aligned}$$

The Laplace Transform of $\sin(t)$ II—An Example

[2]The limit of $f(t)$ as t tends to zero from the right is the value to which $f(t)$ tends as t approaches zero through the positive numbers. In many cases, we assume that $f(t) = 0$ for $t \leq 0$. Sometimes there is a jump in the value of the function at $t = 0$. As the zero value for $t \leq 0$ is often something we do not want to relate to, we sometimes consider only the limit from the right. The limit as one approaches a number, a, from the right is denoted by a^+. By convention $f(0^+) \equiv \lim_{t\to 0^+} f(t)$. Of course, if $f(t)$ is continuous at 0, then $f(0^+) = f(0)$.

We now calculate the Laplace transform of $\sin(t)$ a second way. Let $f(t) = \sin(t)$. Note that $f''(t) = -f(t)$ and that $f(0) = 0, f'(0) = 1$. We find that:

$$\begin{aligned}
\mathcal{L}(-\sin(t))(s) &= s^2\mathcal{L}(\sin(t))(s) - s0 - 1 \Leftrightarrow \\
-\mathcal{L}(\sin(t))(s) &= s^2\mathcal{L}(\sin(t))(s) - 1 \Leftrightarrow \\
(s^2+1)\mathcal{L}(\sin(t))(s) &= 1 \Leftrightarrow \\
\mathcal{L}(\sin(t))(s) &= \frac{1}{s^2+1}.
\end{aligned}$$

The Laplace Transform of $\cos(t)$—An Example

From the fact that $\cos(t) = (\sin(t))'$ and that $\sin(0) = 0$, we see that:

$$\mathcal{L}(\cos(t))(s) = s\mathcal{L}(\sin(t))(s) - 0 = \frac{s}{s^2+1}.$$

An easy corollary of Theorem 2 is:

Corollary 3 $\mathcal{L}\left(\int_0^t f(y)\,dy\right)(s) = \frac{F(s)}{s}$.

PROOF: Let $g(t) = \int_0^t f(y)\,dy$. Clearly, $g(0) = 0$, and $g'(t) = f(t)$. From Theorem 2 we see that $\mathcal{L}(g'(t))(s) = s\mathcal{L}(g(t))(s) - 0 = \mathcal{L}(f(t))(s)$. We find that $\mathcal{L}(\int_0^t f(y)\,dy) = F(s)/s$.

We have seen how to calculate the transform of the derivative of a function; the transform of the derivative is s times the transform of the original function less a constant. We now show that the derivative of the transform of a function is the transform of $-t$ times the original function. By linearity this is identical to:

Theorem 4

$$\mathcal{L}(tf(t))(s) = -\frac{d}{ds}F(s)$$