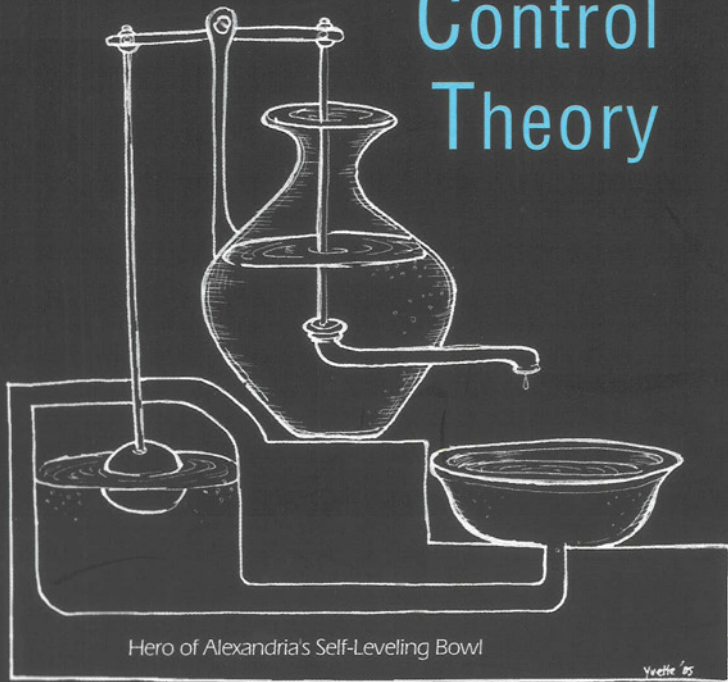


Shlomo Engelberg

VOL. 2

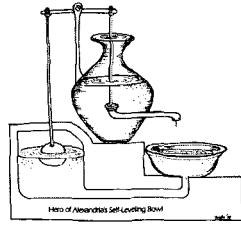
SERIES IN ELECTRICAL AND
COMPUTER ENGINEERING

A Mathematical Introduction to Control Theory



Hero of Alexandria's Self-Leveling Bowl

Imperial College Press



A Mathematical Introduction to Control Theory

SERIES IN ELECTRICAL AND COMPUTER ENGINEERING

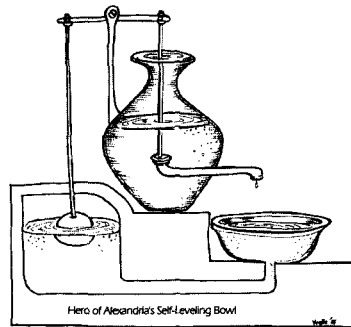
Editor: Wai-Kai Chen (*University of Illinois, Chicago, USA*)

Published:

Vol. 1: **Net Theory and Its Applications**
Flows in Networks
by W. K. Chen

Vol. 2: **A Mathematical Introduction to Control Theory**
by S. Engelberg

A Mathematical Introduction to Control Theory



Shlomo Engelberg
Jerusalem College of Technology, Israel

Published by

Imperial College Press
57 Shelton Street
Covent Garden
London WC2H 9HE

Distributed by

World Scientific Publishing Co. Pte. Ltd.
5 Toh Tuck Link, Singapore 596224
USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601
UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

British Library Cataloguing-in-Publication Data

A catalogue record for this book is available from the British Library.

Series in Electrical and Computer Engineering – Vol. 2

A MATHEMATICAL INTRODUCTION TO CONTROL THEORY

Copyright © 2005 by Imperial College Press

All rights reserved. This book, or parts thereof, may not be reproduced in any form or by any means, electronic or mechanical, including photocopying, recording or any information storage and retrieval system now known or to be invented, without written permission from the Publisher.

For photocopying of material in this volume, please pay a copying fee through the Copyright Clearance Center, Inc., 222 Rosewood Drive, Danvers, MA 01923, USA. In this case permission to photocopy is not required from the publisher.

MATLAB® is a trademark of The MathWorks, Inc. and is used with permission. The MathWorks does not warrant the accuracy of the text or exercises in this book. This book's use or discussion of MATLAB® software or related products does not constitute endorsement or sponsorship by The MathWorks of a particular pedagogical approach or particular use of the MATLAB® software.

ISBN 1-86094-570-8

Printed in Singapore by B & JO Enterprise

Dedication

This book is dedicated to the memory of my beloved uncle
Stephen Aaron Engelberg (1940-2005)
who helped teach me how a *mensh* behaves
and how a person can love and appreciate learning.
May his memory be a blessing.

Preface

Control theory is largely an application of the theory of complex variables, modern algebra, and linear algebra to engineering. The main question that control theory answers is “given reasonable inputs, will my system give reasonable outputs?” Much of the answer to this question is given in the following pages. There are many books that cover control theory. What distinguishes this book is that it provides a complete introduction to control theory without sacrificing either the intuitive side of the subject or mathematical rigor. This book shows how control theory fits into the worlds of mathematics and engineering.

This book was written for students who have had at least one semester of complex analysis and some acquaintance with ordinary differential equations. Theorems from modern algebra are quoted before use—a course in modern algebra *is not* a prerequisite for this book; a single course in complex analysis is. Additionally, to properly understand the material on modern control a first course in linear algebra is necessary. Finally, sections 5.3 and 6.4 are a bit technical in nature; they can be skipped without affecting the flow of the chapters in which they are contained.

In order to make this book as accessible as possible many footnotes have been added in places where the reader’s background—either in mathematics or in engineering—may not be sufficient to understand some concept or follow some chain of reasoning. The footnotes generally add some detail that is not directly related to the argument being made. Additionally, there are several footnotes that give biographical information about the people whose names appear in these pages—often as part of the name of some technique. We hope that these footnotes will give the reader something of a feel for the history of control theory.

In the first seven chapters of this book classical control theory is de-

veloped. The next three chapters constitute an introduction to three important areas of control theory: nonlinear control, modern control, and the control of hybrid systems. The final chapter contains solutions to some of the exercises. The first seven chapters can be covered in a reasonably paced one semester course. To cover the whole book will probably take most students and instructors two semesters.

The first chapter of this book is an introduction to the Laplace transform, a brief introduction to the notion of stability, and a short introduction to MATLAB. MATLAB is used throughout this book as a very fancy calculator. MATLAB allows students to avoid some of the work that would once have had to be done by hand but which cannot be done by a person with either the speed or the accuracy with which a computer can do the same work.

The second chapter bridges the gap between the world of mathematics and of engineering. In it we present transfer functions, and we discuss how to use and manipulate block diagrams. The discussion is in sufficient depth for the non-engineer, and is hopefully not too long for the engineering student who may have been exposed to some of the material previously.

Next we introduce feedback systems. We describe how one calculates the transfer function of a feedback system. We provide a number of examples of how the overall transfer function of a system is calculated. We also discuss the sensitivity of feedback systems to their components. We discuss the conditions under which feedback control systems track their input. Finally we consider the effect of the feedback connection on the way the system deals with noise.

The next chapter is devoted to the Routh-Hurwitz Criterion. We state and prove the Routh-Hurwitz theorem—a theorem which gives a necessary and sufficient condition for the zeros of a real polynomial to be in the left half plane. We provide a number of applications of the theorem to the design of control systems.

In the fifth chapter, we cover the principle of the argument and its consequences. We start the chapter by discussing and proving the principle of the argument. We show how it leads to a graphical method—the Nyquist plot—for determining the stability of a system. We discuss low-pass systems, and we introduce the Bode plots and show how one can use them to determine the stability of such systems. We discuss the gain and phase margins and some of their limitations.

In the sixth chapter, we discuss the root locus diagram. Having covered a large portion of the classical frequency domain techniques for analyz-

ing and designing feedback systems, we turn our attention to time-domain based approaches. We describe how one plots a root locus diagram. We explain the mathematics behind this plot—how the properties of the plot are simply properties of quotients of polynomials with real coefficients. We explain how one uses a root locus plot to analyze and design feedback systems.

In the seventh chapter we describe how one designs compensators for linear systems. Having devoted five chapters largely to the analysis of systems, in this chapter we concentrate on how to design systems. We discuss how one can use various types of compensators to improve the performance of a given system. In particular, we discuss phase-lag, phase-lead, lag-lead and PID (position integral derivative) controllers and how to use them.

In the eighth chapter we discuss nonlinear systems, limit cycles, the describing function technique, and Tsypkin's method. We show how the describing function is a very natural, albeit not always a very good, way of analyzing nonlinear circuits. We describe how one uses it to predict the existence and stability of limit cycles. We point out some of the limitations of the technique. Then we present Tsypkin's method which is an exact method but which is only useful for predicting the existence of limit cycles in a rather limited class of systems.

In the ninth chapter we consider modern control theory. We review the necessary background from linear algebra, and we carefully explain controllability and observability. Then we give necessary and sufficient conditions for controllability and observability of single-input single-output system. We also discuss the pole placement problem.

In the tenth chapter we consider discrete-time control theory and the control of hybrid systems. We start with the necessary background about the z-transform. Then we show how to analyze discrete-time system. The role of the unit circle is described, and the bilinear transform is carefully explained. We describe how to design compensators for discrete-time systems, and we give a brief introduction to the modified z-transform.

In the final chapter we provide solutions to selected exercises. The solutions are generally done at sufficient length that the student will not have to struggle too much to understand them. It is hoped that these solutions will be used *instead of going to a friend or teacher to check one's answer*. They should not be used to avoid thinking about how to go about solving the exercise or to avoid the real work of calculating the solution. In order to develop a good grasp of control theory, one must do problems. It

is not enough to “understand” the material that has been presented; one must *experience* it.

Having spent many years preparing this book and having been helped by many people with this book, I have many people to thank. I am particularly grateful to Professors Richard G. Costello, Jonathan Goodman, Steven Schochet, and Aryeh Weiss who each read this work, critiqued it, and helped me improve it. I also grateful to the many anonymous referees whose comments helped me to improve my presentation of the beautiful results herein described.

I am happy to acknowledge Professor George Anastassiou’s support. Professor Anastassiou has both encouraged me in my efforts to have this work published and has helped me in my search for a suitable publisher. My officemate, Aharon Naiman, has earned my thanks many, many times; he has helped me become more proficient in my use of LaTeX, put up with my enthusiasms, and helped me clarify my thoughts on many points.

My wife, Yvette, and my children, Chananel, Nediva, and Oriya, have always been supportive of my efforts; without Yvette’s support this book would not have been written. My students been kind enough to put up with my penchant for handing out notes in English without complaining too bitterly; their comments have helped improve this book in many ways. My parents have, as always, been pillars of support. Without my father’s love and appreciation of mathematics and science and my mother’s love of good writing I would neither have desired to nor been suited to write a book of this nature. Because of the support of my parents, wife, children, colleagues, and students, writing this book has been a pleasant and meaningful as well as an interesting and challenging experience.

Though all of the many people who have helped and supported me over the years have made their mark on this work I, stubborn as ever, made the final decisions as to what material to include and how to present that material. The nicely turned phrase may well have been provided by a friend or mentor, by a parent or colleague; the mistakes are my own.

Shlomo Engelberg
Jerusalem, Israel

Contents

<i>Preface</i>	vii
1. Mathematical Preliminaries	1
1.1 An Introduction to the Laplace Transform	1
1.2 Properties of the Laplace Transform	2
1.3 Finding the Inverse Laplace Transform	15
1.3.1 Some Simple Inverse Transforms	16
1.3.2 The Quadratic Denominator	18
1.4 Integro-Differential Equations	20
1.5 An Introduction to Stability	25
1.5.1 Some Preliminary Manipulations	25
1.5.2 Stability	26
1.5.3 Why We Obsess about Stability	28
1.5.4 The Tacoma Narrows Bridge—a Brief Case History	29
1.6 MATLAB	29
1.6.1 Assignments	29
1.6.2 Commands	31
1.7 Exercises	32
2. Transfer Functions	35
2.1 Transfer Functions	35
2.2 The Frequency Response of a System	37
2.3 Bode Plots	40
2.4 The Time Response of Certain “Typical” Systems	42
2.4.1 First Order Systems	43
2.4.2 Second Order Systems	44

2.5	Three Important Devices and Their Transfer Functions	46
2.5.1	The Operational Amplifier (op amp)	46
2.5.2	The DC Motor	49
2.5.3	The “Simple Satellite”	50
2.6	Block Diagrams and How to Manipulate Them	51
2.7	A Final Example	54
2.8	Exercises	57
3.	Feedback—An Introduction	61
3.1	Why Feedback—A First View	61
3.2	Sensitivity	62
3.3	More about Sensitivity	64
3.4	A Simple Example	65
3.5	System Behavior at DC	66
3.6	Noise Rejection	70
3.7	Exercises	71
4.	The Routh-Hurwitz Criterion	75
4.1	Proof and Applications	75
4.2	A Design Example	84
4.3	Exercises	87
5.	The Principle of the Argument and Its Consequences	91
5.1	More about Poles in the Right Half Plane	91
5.2	The Principle of the Argument	92
5.3	The Proof of the Principle of the Argument	93
5.4	How are Encirclements Measured?	95
5.5	First Applications to Control Theory	98
5.6	Systems with Low-Pass Open-Loop Transfer Functions	100
5.7	MATLAB and Nyquist Plots	106
5.8	The Nyquist Plot and Delays	107
5.9	Delays and the Routh-Hurwitz Criterion	111
5.10	Relative Stability	113
5.11	The Bode Plots	118
5.12	An (Approximate) Connection between Frequency Specifications and Time Specification	119
5.13	Some More Examples	122
5.14	Exercises	126

6.	The Root Locus Diagram	131
6.1	The Root Locus—An Introduction	131
6.2	Rules for Plotting the Root Locus	133
6.2.1	The Symmetry of the Root Locus	133
6.2.2	Branches on the Real Axis	134
6.2.3	The Asymptotic Behavior of the Branches	135
6.2.4	Departure of Branches from the Real Axis	138
6.2.5	A “Conservation Law”	143
6.2.6	The Behavior of Branches as They Leave Finite Poles or Enter Finite Zeros	144
6.2.7	A Group of Poles and Zeros Near the Origin	145
6.3	Some (Semi-)Practical Examples	147
6.3.1	The Effect of Zeros in the Right Half-Plane	147
6.3.2	The Effect of Three Poles at the Origin	148
6.3.3	The Effect of Two Poles at the Origin	150
6.3.4	Variations on Our Theme	150
6.3.5	The Effect of a Delay on the Root Locus Plot	153
6.3.6	The Phase-lock Loop	156
6.3.7	Sounding a Cautionary Note—Pole-Zero Cancellation	159
6.4	More on the Behavior of the Roots of $Q(s)/K + P(s) = 0$	161
6.5	Exercises	163
7.	Compensation	167
7.1	Compensation—An Introduction	167
7.2	The Attenuator	167
7.3	Phase-Lag Compensation	168
7.4	Phase-Lead Compensation	175
7.5	Lag-lead Compensation	180
7.6	The PID Controller	181
7.7	An Extended Example	188
7.7.1	The Attenuator	189
7.7.2	The Phase-Lag Compensator	189
7.7.3	The Phase-Lead Compensator	191
7.7.4	The Lag-Lead Compensator	193
7.7.5	The PD Controller	195
7.8	Exercises	196

8.	Some Nonlinear Control Theory	203
8.1	Introduction	203
8.2	The Describing Function Technique	204
8.2.1	The Describing Function Concept	204
8.2.2	Predicting Limit Cycles	207
8.2.3	The Stability of Limit Cycles	208
8.2.4	More Examples	211
8.2.4.1	A Nonlinear Oscillator	211
8.2.4.2	A Comparator with a Dead Zone	212
8.2.4.3	A Simple Quantizer	213
8.2.5	Graphical Method	214
8.3	Tsyppkin's Method	216
8.4	The Tsyppkin Locus and the Describing Function Technique	221
8.5	Exercises	223
9.	An Introduction to Modern Control	227
9.1	Introduction	227
9.2	The State Variables Formalism	227
9.3	Solving Matrix Differential Equations	229
9.4	The Significance of the Eigenvalues of the Matrix	230
9.5	Understanding Homogeneous Matrix Differential Equations	232
9.6	Understanding Inhomogeneous Equations	233
9.7	The Cayley-Hamilton Theorem	234
9.8	Controllability	235
9.9	Pole Placement	236
9.10	Observability	237
9.11	Examples	238
9.11.1	Pole Placement	238
9.11.2	Adding an Integrator	240
9.11.3	Modern Control Using MATLAB	241
9.11.4	A System that is not Observable	242
9.11.5	A System that is neither Observable nor Controllable	244
9.12	Converting Transfer Functions to State Equations	245
9.13	Some Technical Results about Series of Matrices	246
9.14	Exercises	248
10.	Control of Hybrid Systems	251

10.1	Introduction	251
10.2	The Definition of the Z-Transform	251
10.3	Some Examples	252
10.4	Properties of the Z-Transform	253
10.5	Sampled-data Systems	257
10.6	The Sample-and-Hold Element	258
10.7	The Delta Function and its Laplace Transform	260
10.8	The Ideal Sampler	261
10.9	The Zero-Order Hold	261
10.10	Calculating the Pulse Transfer Function	262
10.11	Using MATLAB to Perform the Calculations	266
10.12	The Transfer Function of a Discrete-Time System	268
10.13	Adding a Digital Compensator	269
10.14	Stability of Discrete-Time Systems	271
10.15	A Condition for Stability	273
10.16	The Frequency Response	276
10.17	A Bit about Aliasing	278
10.18	The Behavior of the System in the Steady-State	278
10.19	The Bilinear Transform	279
10.20	The Behavior of the Bilinear Transform as $T \rightarrow 0$	284
10.21	Digital Compensators	285
10.22	When Is There No Pulse Transfer Function?	288
10.23	An Introduction to the Modified Z-Transform	289
10.24	Exercises	291
11.	Answers to Selected Exercises	295
11.1	Chapter 1	295
11.1.1	Problem 1	295
11.1.2	Problem 3	296
11.1.3	Problem 5	297
11.1.4	Problem 7	298
11.2	Chapter 2	298
11.2.1	Problem 1	298
11.2.2	Problem 3	299
11.2.3	Problem 5	300
11.2.4	Problem 7	301
11.3	Chapter 3	303
11.3.1	Problem 1	303
11.3.2	Problem 3	304

	11.3.3	Problem 5	304
	11.3.4	Problem 7	305
11.4	Chapter 4		305
	11.4.1	Problem 1	305
	11.4.2	Problem 3	306
	11.4.3	Problem 5	307
	11.4.4	Problem 7	307
	11.4.5	Problem 9	309
11.5	Chapter 5		310
	11.5.1	Problem 1	310
	11.5.2	Problem 3	311
	11.5.3	Problem 5	311
	11.5.4	Problem 7	312
	11.5.5	Problem 9	314
	11.5.6	Problem 11	315
11.6	Chapter 6		316
	11.6.1	Problem 1	316
	11.6.2	Problem 3	316
	11.6.3	Problem 5	318
	11.6.4	Problem 7	319
	11.6.5	Problem 9	320
11.7	Chapter 7		322
	11.7.1	Problem 1	322
	11.7.2	Problem 3	324
	11.7.3	Problem 5	326
	11.7.4	Problem 7	327
	11.7.5	Problem 9	330
11.8	Chapter 8		332
	11.8.1	Problem 1	332
	11.8.2	Problem 3	335
	11.8.3	Problem 5	336
	11.8.4	Problem 7	337
11.9	Chapter 9		337
	11.9.1	Problem 6	337
	11.9.2	Problem 7	338
11.10	Chapter 10		339
	11.10.1	Problem 4	339
	11.10.2	Problem 10	339
	11.10.3	Problem 13	340

11.10.4 Problem 16	342
11.10.5 Problem 17	343
11.10.6 Problem 19	343

<i>Bibliography</i>	345
---------------------	-----

<i>Index</i>	347
--------------	-----

Chapter 1

Mathematical Preliminaries

1.1 An Introduction to the Laplace Transform

Much of this chapter is devoted to describing and deriving some of the properties of the one-sided Laplace transform. The Laplace transform is the engineer's most important tool for analyzing the stability of linear, time-invariant, continuous-time systems. The Laplace transform is defined as:

$$\mathcal{L}(f(t))(s) \equiv \int_0^{\infty} e^{-st} f(t) dt.$$

We often write $F(s)$ for the Laplace transform of $f(t)$. It is customary to use lower-case letters for functions of time, t , and to use the same letter—but in its upper-case form—for the Laplace transform of the function; throughout this book, we follow this practice.

We assume that the functions $f(t)$ are of *exponential type*—that they satisfy an inequality of the form $|f(t)| \leq Ce^{\alpha t}$, $C \in \mathcal{R}$. If the real part of s , $\Re(s)$, satisfies $\Re(s) < -\alpha$, then the integral that defines the Laplace transform converges. The Laplace transform's usefulness comes largely from the fact that it allows us to convert differential and integro-differential equations into algebraic equations.

We now calculate the Laplace transform of some functions. We start with the unit step function (also known as the Heaviside ¹ function):

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}.$$

¹After Oliver Heaviside (1850-1925) who between 1880 and 1887 invented the “operational calculus” [OR]. His operational calculus was widely used in its time. The Laplace transform that is used today is a “cousin” of Heaviside's operational calculus [Dea97].

From the definition of the Laplace transform, we find that:

$$\begin{aligned} U(s) &= \mathcal{L}(u(t))(s) \\ &= \int_0^{\infty} e^{-st} \cdot 1 \, dt \\ &= \left. \frac{e^{-st}}{-s} \right|_0^{\infty} \\ &= \lim_{t \rightarrow \infty} \frac{e^{-st}}{-s} - \frac{1}{-s}. \end{aligned}$$

Denote the real part of s by α and its imaginary part by β . Continuing our calculation, we find that:

$$\begin{aligned} U(s) &= \lim_{t \rightarrow \infty} e^{-\alpha t} \frac{e^{-j\beta t}}{-s} + \frac{1}{s} \\ &= 0 + \frac{1}{s} = \frac{1}{s}. \end{aligned}$$

This holds as long as $\alpha > 0$. In this case the first term in the limit:

$$\lim_{t \rightarrow \infty} e^{-\alpha t} \frac{e^{-j\beta t}}{-s}$$

is approaching zero while the second term—though oscillatory—is bounded. In general, we assume that s is chosen so that integrals and limits that must converge do. For our purposes, the region of convergence (in terms of s) of the integral is not terribly important.

Next we consider $\mathcal{L}(e^{at})(s)$. We find that:

$$\begin{aligned} \mathcal{L}(e^{at})(s) &= \int_0^{\infty} e^{-st} e^{at} \, dt \\ &= \left. \frac{e^{(a-s)t}}{a-s} \right|_0^{\infty} \\ &= \frac{1}{s-a}. \end{aligned}$$

1.2 Properties of the Laplace Transform

The first property of the Laplace transform is its *linearity*.

Theorem 1

$$\mathcal{L}(\alpha f(t) + \beta g(t))(s) = \alpha F(s) + \beta G(s).$$

Simply put, “the Laplace transform of a linear combination is the linear combination of the Laplace transforms.”

PROOF: Making use of the properties of the integral, we find that:

$$\begin{aligned}\mathcal{L}(\alpha f(t) + \beta g(t))(s) &= \int_0^{\infty} e^{-st} (\alpha f(t) + \beta g(t)) dt \\ &= \alpha \int_0^{\infty} e^{-st} f(t) dt + \beta \int_0^{\infty} e^{-st} g(t) dt \\ &= \alpha F(s) + \beta G(s).\end{aligned}$$

We see that the linearity of the Laplace transform is part of its “inheritance” from the integral which defines it.

The Laplace Transform of $\sin(t)$ I—An Example

Following the engineering convention that $j \equiv \sqrt{-1}$, we write:

$$\sin(t) = \frac{e^{jt} - e^{-jt}}{2j}.$$

By linearity we find that:

$$\mathcal{L}(\sin(t))(s) = \frac{1}{2j} (\mathcal{L}(e^{jt})(s) - \mathcal{L}(e^{-jt})(s)).$$

Making use of the fact that we know what the Laplace transform of an exponential is, we find that:

$$\mathcal{L}(\sin(t))(s) = \frac{1}{2j} \left(\frac{1}{s-j} - \frac{1}{s+j} \right) = \frac{1}{s^2+1}.$$

The next property we consider is the property that makes the Laplace transform so useful. As we shall see, it is possible to calculate the Laplace transform of the solution of a constant-coefficient ordinary differential equation (ODE) *without solving the ODE*.

Theorem 2 Assume that $f(t)$ has a well defined limit as t approaches zero from the right². Then we find that:

$$\mathcal{L}(f'(t))(s) = sF(s) - f(0^+).$$

PROOF: This result is proved by making use of integration by parts. We see that:

$$\mathcal{L}(f'(t))(s) = \int_0^{\infty} e^{-st} f'(t) dt.$$

Let $u = e^{-st}$ and $dv = f'(t)dt$. Then $du = -se^{-st}$ and $v = f(t)$. Assuming that $\alpha = \mathcal{R}(s) > 0$, we find that:

$$\begin{aligned} \int_0^{\infty} e^{-st} f'(t) dt &= - \int_0^{\infty} \frac{d}{dt} e^{-st} f(t) dt + e^{-st} f(t) \Big|_0^{\infty} \\ &= s \int_0^{\infty} e^{-st} f(t) dt + \lim_{t \rightarrow \infty} e^{-st} f(t) - f(0^+) \\ &= sF(s) + 0 - f(0^+) \\ &= sF(s) - f(0^+). \end{aligned}$$

We take the limit of $f(t)$ as $t \rightarrow 0^+$ because the integral itself deals only with positive values of t . Often we dispense with the added generality that the limit from the right gives us, and we write $f(0)$.

We can use this theorem to find the Laplace transform of the second (or higher) derivative of a function. To find the Laplace transform of the second derivative of a function, one applies the theorem twice. I.e.:

$$\begin{aligned} \mathcal{L}(f''(t))(s) &= s\mathcal{L}(f'(t))(s) - f'(0) \\ &= s(sF(s) - f(0)) - f'(0) \\ &= s^2F(s) - sf'(0) - f(0). \end{aligned}$$

The Laplace Transform of $\sin(t)$ II—An Example

²The limit of $f(t)$ as t tends to zero from the right is the value to which $f(t)$ tends as t approaches zero through the positive numbers. In many cases, we assume that $f(t) = 0$ for $t \leq 0$. Sometimes there is a jump in the value of the function at $t = 0$. As the zero value for $t \leq 0$ is often something we do not want to relate to, we sometimes consider only the limit from the right. The limit as one approaches a number, a , from the right is denoted by a^+ . By convention $f(0^+) \equiv \lim_{t \rightarrow 0^+} f(t)$. Of course, if $f(t)$ is continuous at 0, then $f(0^+) = f(0)$.

We now calculate the Laplace transform of $\sin(t)$ a second way. Let $f(t) = \sin(t)$. Note that $f''(t) = -f(t)$ and that $f(0) = 0, f'(0) = 1$. We find that:

$$\begin{aligned}\mathcal{L}(-\sin(t))(s) &= s^2\mathcal{L}(\sin(t))(s) - s \cdot 0 - 1 \Leftrightarrow \\ -\mathcal{L}(\sin(t))(s) &= s^2\mathcal{L}(\sin(t))(s) - 1 \Leftrightarrow \\ (s^2 + 1)\mathcal{L}(\sin(t))(s) &= 1 \Leftrightarrow \\ \mathcal{L}(\sin(t))(s) &= \frac{1}{s^2 + 1}.\end{aligned}$$

The Laplace Transform of $\cos(t)$ —An Example

From the fact that $\cos(t) = (\sin(t))'$ and that $\sin(0) = 0$, we see that:

$$\mathcal{L}(\cos(t))(s) = s\mathcal{L}(\sin(t))(s) - 0 = \frac{s}{s^2 + 1}.$$

An easy corollary of Theorem 2 is:

Corollary 3 $\mathcal{L}\left(\int_0^t f(y) dy\right)(s) = \frac{F(s)}{s}.$

PROOF: Let $g(t) = \int_0^t f(y) dy$. Clearly, $g(0) = 0$, and $g'(t) = f(t)$. From Theorem 2 we see that $\mathcal{L}(g'(t))(s) = s\mathcal{L}(g(t))(s) - 0 = \mathcal{L}(f(t))(s)$. We find that $\mathcal{L}\left(\int_0^t f(y) dy\right) = F(s)/s$.

We have seen how to calculate the transform of the derivative of a function; the transform of the derivative is s times the transform of the original function less a constant. We now show that the derivative of the transform of a function is the transform of $-t$ times the original function. By linearity this is identical to:

Theorem 4

$$\mathcal{L}(tf(t))(s) = -\frac{d}{ds}F(s)$$

sample content of A Mathematical Introduction to Control Theory (Series in Electrical and Computer Engineering)

- [Object-Oriented Software Engineering Using UML, Patterns, and Java \(3rd Edition\) for free](#)
- [Act of Treason \(Mitch Rapp, Book 9\) pdf, azw \(kindle\)](#)
- [The Knitting Circle Rapist Annihilation Squad pdf, azw \(kindle\), epub, doc, mobi](#)
- [download online The Song House pdf, azw \(kindle\)](#)

- <http://ramazotti.ru/library/Il-destino-della-tecnica--BUR-Saggi-.pdf>
- <http://growingsomeroots.com/ebooks/Pacific-Edge--Three-Californias-Triptych--Book-3-.pdf>
- <http://betsy.wesleychapelcomputerrepair.com/library/Antikkabinettet.pdf>
- <http://transtrade.cz/?ebooks/Utah-s-National-Parks--Hiking--Camping--and-Vacationing-in-Utah-s-Canyon-Country---Zion--Bryce--Capitol-Reef--A>